Sub-homogeneous Positive Systems are Insensitive to Heterogeneous Time-Varying Delays

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Positive systems

Wide variety of applications, including

- **Social science**: population models, etc.,
- **Biology/Medicine**: reaction dynamics, proteins, etc.,
- **Economy**: stochastic models, markov jump systems, etc.

\[
\dot{p}_i(t) = -p_i(t) + \mathcal{I}_i(p(t))
\]

\[
p_i(t) \geq 0, \quad t \geq 0
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**Positive systems**

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Positive systems

Why heterogeneous time delays?

Omnipresent in distributed positive systems

Example: power control for wireless networks,

\[ \dot{p}_i(t) = -p_i(t) + I_i(p_1(t - \tau^i_1(t)), \ldots, p_n(t - \tau^i_n(t))) \]
Monotone systems with heterogeneous delays

Consider the continuous-time nonlinear system

\[
\begin{align*}
\dot{x}_i(t) &= f_i(x(t)) + g_i(x_1(t - \tau^i_1(t)), \ldots, x_n(t - \tau^i_n(t))), \quad t \geq 0, \\
x_i(t) &= \varphi_i(t), \quad t \in [-\tau_{\max}, 0],
\end{align*}
\]

where \( \varphi_i(t) : [-\tau_{\max}, 0] \rightarrow \mathbb{R} \) is the initial condition.

The delays are continuous functions of time, but otherwise arbitrarily varying.

We assume that \( 0 \leq \tau^i_j(t) \leq \tau_{\max} \), and allow \( \tau_{\max} \rightarrow \infty \).

Definition. System is called monotone if

\[
\varphi(t) \geq \varphi'(t) \Rightarrow x(t, \varphi) \geq x(t, \varphi'), \quad \forall t \geq 0
\]
Monotone systems with heterogeneous delays

Consider the continuous-time nonlinear system

\[
\dot{x}_i(t) = f_i(x(t)) + g_i(x_1(t - \tau_{1i}(t)), \ldots, x_n(t - \tau_{ni}(t))), \quad t \geq 0,
\]

\[
x_i(t) = \varphi_i(t), \quad t \in [-\tau_{\text{max}}, 0],
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We assume that \( 0 \leq \tau_{ji}(t) \leq \tau_{\text{max}}, \) and allow \( \tau_{\text{max}} \to \infty. \)

**Definition.** System is called **monotone** if

\[
\varphi(t) \geq \varphi'(t) \implies x(t, \varphi) \geq x(t, \varphi'), \quad \forall t \geq 0
\]
Fact. System

\[
\dot{x}_i(t) = f_i(x(t)) + g_i(x_1(t - \tau_1^i(t)), \ldots, x_n(t - \tau_n^i(t))), \quad t \geq 0, \\
x_i(t) = \varphi_i(t), \quad t \in [-\tau_{\text{max}}, 0],
\]

is monotone if \( f \) is cooperative, i.e.,

\[
\frac{\partial f_i}{\partial x_j}(x) \geq 0, \quad i \neq j, \quad \forall x \in \mathbb{R}_+^n,
\]

and \( g \) is non-decreasing, i.e.,

\[
x \geq y \implies g(x) \geq g(y)
\]

Positive monotone systems with heterogeneous delays

**Definition.** System

\[
\begin{align*}
\dot{x}_i(t) &= f_i(x(t)) + g_i(x_1(t - \tau^i_1(t)), \ldots, x_n(t - \tau^i_n(t))), \quad t \geq 0, \\
x_i(t) &= \varphi_i(t), \quad t \in [-\tau_{\text{max}}, 0],
\end{align*}
\]

is called **positive** if positive orthant is forward invariant:

\[
\varphi(\cdot) \geq 0 \Rightarrow x(t, \varphi) \geq 0, \quad \forall t \geq 0
\]

**Proposition.** Monotone system is positive if and only if

\[
f(0) + g(0) \geq 0
\]

**Fact.** Some positive monotone systems remain asymptotically stable under constant time delays!

- homogeneous positive monotone systems [Mason et al, 2009],
- sub-homogeneous positive monotone systems [Bokharaie et al, 2012].
Positive monotone systems with heterogeneous delays

Definition. System

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\begin{align*}
\dot{x}_i(t) &= f_i(x(t)) + g_i(x_1(t - \tau_1^i(t)), \ldots, x_n(t - \tau_n^i(t))), \quad t \geq 0, \\
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Our contributions

Does this insensitivity property hold also for *heterogeneous time-varying* delays?

1. Establish necessary and sufficient conditions for delay-independent stability of positive monotone systems with heterogeneous time-varying delays,

2. Show the insensitivity of sub-homogeneous positive monotone systems to heterogeneous time-varying delays.
Our contributions

Does this insensitivity property hold also for *heterogeneous time-varying* delays?

1. Establish necessary and sufficient conditions for delay-independent stability of positive monotone systems with heterogeneous time-varying delays,

2. Show the insensitivity of sub-homogeneous positive monotone systems to heterogeneous time-varying delays.
Asymptotic stability under constant time delays

**Theorem** [Smith, 1995]

For the monotone system with constant delays

\[
\dot{x}(t) = f(x(t)) + g(x(t - \tau_{\text{max}}))
\]

- if there exists a vector \( v \) such that
  \[
f(v) + g(v) \leq 0
  \]
  the trajectory \( x(t, v) \) is non-increasing.
- if there exists a vector \( w \) such that
  \[
f(w) + g(w) \geq 0
  \]
  the trajectory \( x(t, w) \) is non-decreasing.
Asymptotic stability under constant time delays

Equilibrium \( x^* \in [w, v] \) is asymptotically stable for initial conditions

\[
w \leq \varphi(t) \leq v, \quad \forall t \in [-\tau_{\text{max}}, 0]
\]

For time-varying delays, this result does not hold!
Theorem. Consider two monotone systems

\[
\dot{x}_i(t) = f_i(x(t)) + g_i(x_1(t - \tau^i_1(t)), \ldots, x_n(t - \tau^i_n(t))), \\
\dot{y}_i(t) = f_i(y(t)) + g_i(y(t - \tau_{\text{max}})),
\]

• If there exists a vector \( v \) such that

\[
f(v) + g(v) \leq 0
\]

then \( x(t, v) \leq y(t, v) \).

• If there exists a vector \( w \) such that

\[
f(w) + g(w) \geq 0
\]

then \( x(t, w) \geq y(t, w) \).
Asymptotic stability under heterogeneous time-varying delays

For all bounded heterogeneous time-varying delays, \( x^* \in [w, v] \) is asymptotically stable with respect to initial conditions

\[
w \leq \varphi(t) \leq v, \quad \forall t \in [-\tau_{\text{max}}, 0]
\]
A vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) is called **sub-homogeneous** of degree \( \alpha > 0 \) if

\[
f(\lambda x) \leq \lambda^\alpha f(x), \quad \forall x \in \mathbb{R}^n, \forall \lambda \geq 1
\]

- includes linear mappings \( f(x) = Ax \) and homogeneous vector fields

\[
f(\lambda x) = \lambda^\alpha f(x), \quad \forall x \in \mathbb{R}^n, \forall \lambda > 0
\]

When \( f \) and \( g \) are sub-homogeneous, the positive monotone system

\[
\begin{align*}
\dot{x}_i(t) & = f_i(x(t)) + g_i(x_1(t - \tau^i_1(t)), \ldots, x_n(t - \tau^i_n(t))), & t \geq 0, \\
x_i(t) & = \varphi_i(t), & t \in [-\tau_{\text{max}}, 0],
\end{align*}
\]

is called sub-homogeneous positive monotone.
**Theorem.** The sub-homogeneous positive monotone system

\[
\dot{x}_i(t) = f_i(x(t)) + g_i(x_1(t - \tau^i_1(t)), \ldots, x_n(t - \tau^i_n(t))), \quad t \geq 0,
\]

\[
x_i(t) = \varphi_i(t), \quad t \in [-\tau_{\text{max}}, 0]
\]

is globally asymptotically stable if and only if

\[
\dot{x}_i(t) = f_i(x(t)) + g_i(x(t)), \quad t \geq 0
\]

is globally asymptotically stable.
Sub-homogeneous positive monotone systems

Proof idea

Fact

For the monotone system with constant delays

\[ \dot{x}(t) = f(x(t)) + g(x(t - \tau_{\text{max}})) \]

if there exist vectors \( v \) and \( w \) such that \( w \leq v \) and

\[
\begin{align*}
    f(v) + g(v) &\leq 0, \\
    f(w) + g(w) &\geq 0,
\end{align*}
\]

then equilibrium \( x^* \in [w, v] \) is asymptotically stable for initial conditions

\[ w \leq \varphi(t) \leq v, \quad \forall t \in [-\tau_{\text{max}}, 0] \]

Proof idea

- For time-varying delays, this result does not hold!
Sub-homogeneous positive monotone systems

Proof idea

Lemma.

For the monotone system with heterogeneous time-varying delays

\[ \dot{x}_i(t) = f_i(x(t)) + g_i\left(x_1(t - \tau_1^i(t)), \ldots, x_n(t - \tau_n^i(t))\right), \]

if there exist vectors \( v \) and \( w \) such that \( w \leq v \) and

\[ f(v) + g(v) \leq 0, \]
\[ f(w) + g(w) \geq 0, \]

then equilibrium \( x^* \in [w, v] \) is asymptotically stable for initial conditions

\[ w \leq \varphi(t) \leq v, \quad \forall t \in [-\tau_{\text{max}}, 0] \]
Proof idea

If \( \dot{x}_i(t) = f_i(x(t)) + g_i(x(t)) \) is globally asymptotically stable, then, for any \( \varphi(t) \geq 0 \), there exist vectors \( v \) and \( w \) such that

\[
 w \leq \varphi(t) \leq v,
\]

and

\[
 f(v) + g(v) \leq 0, \\
 f(w) + g(w) \geq 0.
\]

According to the previous lemma,

\[
 \lim_{t \to \infty} x(t, \varphi) = x^*
\]
Conclusions

Concluding remarks

Summary
Positive monotone systems under heterogeneous time-varying delays

- Delay-independent condition for asymptotic stability
- Delay-insensitivity of sub-homogeneous positive monotone systems

Future directions

- Stability of positive monotone systems with unbounded time delays
- Insensitivity of more general classes of monotone systems
Thank you!

Questions?