Distributed Minimum-Time Weight Balancing over Digraphs

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Outline

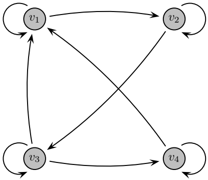
1. Motivation - Introduction
2. Notation and mathematical preliminaries
3. Weight balancing in digraphs
4. Minimum-time weight balancing
5. Examples
6. Concluding remarks

Motivation

- *Applications* where weight balance plays a key role:
  - Synchronization
  - Average consensus via linear iterations (special case of synchronization without dynamics) – applications in multicomponent systems where one is interested in distributively averaging measurements, e.g., sensor networks
  - Traffic-flow problems captured by n junctions and m one-way streets
  - Stable flocking of agents with significant inertial effects
  - Pinning control, optoelectronics, biology, ...
- *Finite-time algorithms* are generally more desirable
  - they converge in finite-time
  - closed-loop systems under finite-time control usually demonstrate better disturbance rejection properties

Distributed system model

- Distributed systems conveniently captured by digraphs
  1. Components represented by vertices (nodes)
  2. Communication and sensing links represented by edges



- Consider a network with nodes $(v_1, v_2, \ldots, v_N)$
  E.g., sensors, robots, unmanned vehicles, resources, etc.

- Nodes can receive information according to (possibly directed) communication links

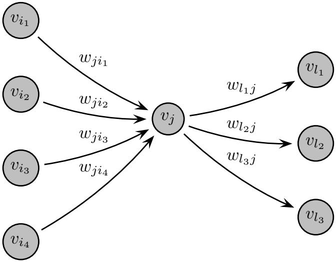
- Each node v_j has some initial value $x_j[0]$ (could be belief, position, velocity, etc.)

Graph notation

- *Digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
  - Nodes (system components) $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$
  - Edges (directed communication links) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ where $(v_j, v_i) \in \mathcal{E}$ iff node v_j can receive information from node v_i
  - In-neighbors $\mathcal{N}_j^- = \{v_i \mid (v_j, v_i) \in \mathcal{E}\}$; in-degree $\mathcal{D}_j^- = |\mathcal{N}_j^-|$
  - Out-neighbors $\mathcal{N}_j^+ = \{v_l \mid (v_l, v_j) \in \mathcal{E}\}$; out-degree $\mathcal{D}_j^+ = |\mathcal{N}_j^+|$
- *Adjacency matrix A:* $A(j, i) = 1$ if $(v_j, v_i) \in \mathcal{E}$; $A(j, i) = 0$ otherwise
- *Undirected graph:* $(v_j, v_i) \in \mathcal{E}$ iff $(v_i, v_j) \in \mathcal{E}$ (bidirectional links)
  In undirected graphs, we have (for each node j)
  $\mathcal{N}_j^+ = \mathcal{N}_j^-$ and $\mathcal{D}_j^+ = \mathcal{D}_j^- = \mathcal{D}_j$; also, $A = A^T$
- *(Strongly) connected (di)graph* if for any $i, j \in \mathcal{V}$, $j \neq i$, there exists a (directed) path connecting them, i.e.,

$$v_i = v_{i_0} \to v_{i_1},\ v_{i_1} \to v_{i_2},\ \ldots,\ v_{i_{t-1}} \to v_{i_t} = v_j$$

Problem formulation



Weight-balanced digraph:

Sum of weights on incoming links = Sum of weights on outgoing links

1. $w_{ji} > 0$ for each edge $(v_j, v_i) \in \mathcal{E}$;
2. $w_{ji} = 0$ if $(v_j, v_i) \notin \mathcal{E}$;
3. $S_j^+ = S_j^- \ \forall \ v_j \in V$, where $S_j^- = \sum_{v_i \in \mathcal{N}_j^-} w_{ji}$, $S_j^+ = \sum_{v_l \in \mathcal{N}_j^+} w_{lj}$

Weight balancing in graphs

- Real-weight balancing:
  - *Asymptotic* weight balancing; no known bound of convergence [C.N.H. & A.R., 2012]
  - *Asymptotic* weight balancing; each agent is assumed to distinguish the information coming from other agents; a global stopping time is set to stop performing the balancing [Priolo *et al*, 2013]
  - *Geometric* convergence rate with known rate of convergence [T.C. & C.N.H., 2013]

- Integer-weight balancing:
  - Finite number of steps; no known bound for convergence [B. Gharesifard and J. Cortés., 2012]
  - Finite number of steps; upper bound of $\mathcal{O}(n^7)$ [Apostolos Rikos, T.C. & C.N.H., 2014]

- *Setting:* Nodes distributively adjust the weights of their outgoing links such that the digraph asymptotically becomes weight-balanced; they observe but cannot set the weights of their incoming links
- Each node v_j initializes the weights of all of its outgoing links to unity, i.e., $w_{lj}[0] = 1$, $\forall v_l \in \mathcal{N}_j^+$ (different initial weights also possible)
- Nodes enter an iterative stage where node v_j performs the following steps:
  1. It computes its weight imbalance defined by
  $$x_j[k] \triangleq S_j^-[k] - S_j^+[k]$$
  2. If $x_j[k]$ is positive (resp. negative), all the weights of its outgoing links are increased (resp. decreased) by an equal amount and proportionally to $x_j[k]$, specifically, $\forall v_l \in \mathcal{N}_j^+$,
  $$w_{lj}[k+1] = w_{lj}[k] + \beta_j \left(\frac{S_j^-[k]}{\mathcal{D}_j^+} - w_{lj}[k] \right), \quad \beta_j \in (0,1) \quad (1)$$

- *Intuition:* we compare $S_j^-[k]$ with $S_j^+[k] = \mathcal{D}_j^+ w_{lj}[k]$. If $S_j^+[k] > S_j^-[k]$ (resp. $S_j^+[k] < S_j^-[k]$), then the algorithm reduces (resp. increases) the weights on the outgoing links

Proposition 1

If a digraph is strongly connected, **the weight balancing algorithm** asymptotically reaches a steady state weight matrix W^* that forms a weight-balanced digraph, with geometric convergence rate equal to $R_\infty(P) = -\ln \delta(P)$, where

$$P_{ji} \triangleq \begin{cases} 1 - \beta_j, & \text{if } i = j, \\ \beta_j / \mathcal{D}_j^+, & \text{if } v_i \in \mathcal{N}_j^-, \end{cases}$$

and $\delta(P) \triangleq \max\{|\lambda| : \lambda \in \sigma(P)), \lambda \neq 1\}$

Distributed *finite-time* methods in graphs

- Finite-time approaches for *undirected* graphs:
  - *Finite-time* average consensus [J. Cortés, 2006], [S. Sundaram & C.N.H., 2007], [Wang & Xiao, 2010]
  - *Minimum-time* average consensus [Y. Yuan *et al*, 2009] (associated with final value of linear iterations)

- Finite-time approaches for *directed* graphs:
  - *Minimum-time* average consensus in digraphs [T.C. *et al*, 2013] (used the same concept for final value of linear iterations)

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We propose an algorithm that combines *asymptotic weight-balancing* with *distributed final value of linear iterations* and has a convergence upper bound $\mathcal{O}(2n)$.

Preliminaries - I

Minimal polynomial of a matrix pair

The minimal polynomial associated with the matrix pair $[P, e_j^T]$, denoted by $q_j(t) = t^{M_j+1} + \sum_{i=0}^{M_j} \alpha_i^{(j)} t^i$, is the monic polynomial of minimum degree $M_j + 1$ that satisfies $e_j^T q_j(P) = 0$.

Easy to show (e.g., using the techniques in [Y. Yuan *et al*, 2009]) that

$$\sum_{i=0}^{M_j+1} \alpha_i^{(j)} w_j[k+i] = 0, \quad \forall k \in \mathbb{Z}_+ \ ,$$

where $\alpha_{M_j+1}^{(j)} = 1$. Denote z-transform of $w_j[k]$ as $W_j(z) \triangleq \mathbb{Z}(w_j[k])$. Then,

$$W_j(z) = \frac{\sum_{i=1}^{M_j+1} \alpha_i^{(j)} \sum_{\ell=0}^{i-1} w_j[\ell] z^{i-\ell}}{q_j(z)} \ ,$$

where $q_j(z)$ is the minimal polynomial of $[P, e_j^T]$.

Preliminaries - II

Define the following polynomial:

$$p_j(z) \triangleq \frac{q_j(z)}{z-1} \triangleq \sum_{i=0}^{M_j} \beta_i^{(j)} z^i$$

The application of the final value theorem (FVT) yields:

$$\phi_w(j) = \lim_{k\to\infty} w_j[k] = \lim_{z\to 1}(z-1)W_j(z) = \frac{\mathbf{w}_{M_j}^\top \boldsymbol{\beta}_j}{\mathbf{1}^\top \boldsymbol{\beta}_j}$$

where

- $\mathbf{w}_{M_j}^\top = (w_j[0], w_j[1], \ldots, w_j[M_j])$
- $\boldsymbol{\beta}_j$ is the vector of coefficients of the polynomial $p_j(z)$

Preliminaries - II

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where

- $\mathbf{w}_{M_j}^\mathsf{T} = (w_j[0], w_j[1], \ldots, w_j[M_j])$
- $\boldsymbol{\beta}_j$ is the vector of coefficients of the polynomial $p_j(z)$

How can we obtain $\boldsymbol{\beta}_j$ in the computation of final values?

Preliminaries - III

- Consider the vectors of differences between $2k+1$ successive discrete-time values of $w_j[k]$ at node v_j and $x_j[k]$:

$$\overline{w}_{2k}^{\mathsf{T}} = (w_j[1] - w_j[0], \ldots, w_j[2k+1] - w_j[2k])$$

- Let us define their associated Hankel matrix:

$$\Gamma\{w_{2k}^{\mathsf{T}}\} \triangleq \begin{bmatrix} w_j[0] & w_j[1] & \ldots & w_j[k] \\ w_j[1] & w_j[2] & \ldots & w_j[k+1] \\ \vdots & \vdots & \ddots & \vdots \\ w_j[k] & w_j[k+1] & \ldots & w_j[2k] \end{bmatrix}$$

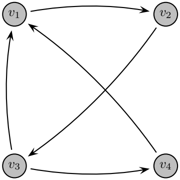
- β_j can be computed as *the kernel of the first defective Hankel matrix* for $\Gamma\{\overline{w}_{2k}^{\mathsf{T}}\}$
- For arbitrary initial conditions w_0, except a set of initial conditions with Lebesgue measure zero.

- **Input:** A strongly connected digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$
- **Data:** Successive observations for $w_j[k]$, $\forall v_j \in \mathcal{V}$ using simultaneous iterations of (1) for *asymptotic weight-balancing* with initial conditions $w[0] = w_0$
- **Step 1:** Each node $v_j \in \mathcal{V}$ stores the vectors of differences $\overline{w}_{M_j}^{\mathsf{T}}$ between successive values of $w_j[k]$
- **Step 2:** Increase the dimension k of $\Gamma\{\overline{w}_{M_j}^{\mathsf{T}}\}$, until it loses rank; store the first defective matrix
- **Step 3:** The kernel $\boldsymbol{\beta}_j = \left(\beta_0, \ldots, \beta_{M_j-1}, 1\right)^{\mathsf{T}}$ of the first defective matrix gives the value ϕ_w which is the final value of iteration (1); i.e.,

$$w_j^* = \phi_w(j) = \frac{w_{M_j}^{\mathsf{T}} \boldsymbol{\beta}_j}{\mathbf{1}^{\mathsf{T}} \boldsymbol{\beta}_j}$$

Illustrative example

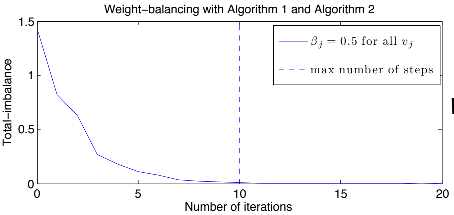
- Example borrowed by [B.Gharesifard & J.Cortés, 2010]



- Concerned with the absolute balance defined as

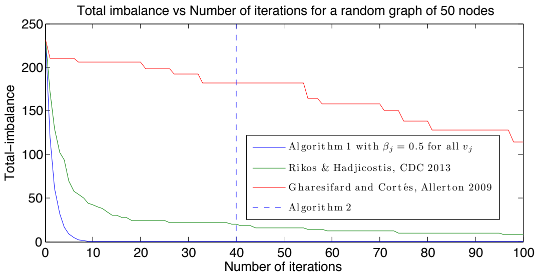
$$\varepsilon[k] = \sum_{j=1}^{n} |x_j[k]|$$

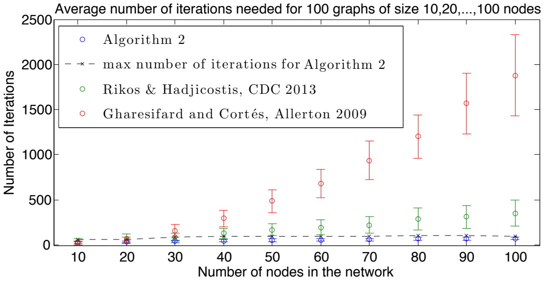
- If weight balance is achieved, then $\varepsilon[k] = 0$ and $x_j[k] = 0, \forall v_j \in \mathcal{V}$



$$W^\star = \begin{bmatrix} 0 & 0 & 0.7143 & 0.7143 \\ 1.4286 & 0 & 0 & 0 \\ 0 & 1.4286 & 0 & 0 \\ 0 & 0 & 0.7143 & 0 \end{bmatrix}$$

Comparisons with other works





Concluding remarks and future directions

Conclusions:

- Proposed a distributed iterative algorithm, in which each node:
  - has knowledge of its *outgoing* links
  - reaches weight balancing in *directed* graphs in *minimum-time*
  - uses only output observations at each component (finite-time history of its own values)

Future work:

- Study weight balancing in a graph with time-varying delays
- Consider noisy output observations

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Thank You!

Questions?

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