Decentralised Minimum-Time Average Consensus in Digraphs

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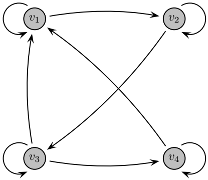
IEEE Conference on Decision and Control (CDC ’13)
Florence, Italy, December 2013

1. Distributed coordination has been the subject of extensive research for a long time
   - parallel and distributed computation
   - distributed optimization in sensor networks
   - formation control of robotic networks
   - dynamics of opinion forming
2. Finite-time algorithms are generally more desirable
   - they converge in finite-time
   - closed-loop systems under finite-time control usually demonstrate better disturbance rejection properties

- Distributed systems conveniently captured by digraphs
    1. Components represented by vertices (nodes)
    2. Communication and sensing links represented by edges



- Consider a network with nodes $(v_1, v_2, \ldots, v_N)$
  E.g., sensors, robots, unmanned vehicles, resources, etc.
- Nodes can receive information according to (possibly directed) communication links
- Each node v_j has some initial value $x_j[0]$ (could be belief, position, velocity, etc.)

Consensus and Average Consensus

- *Typical objective:* Calculate functions of initial values in a distributed manner (e.g., $\max_\ell\{x_\ell[0]\}$, $\sum_\ell x_\ell^2[0]$, etc.)
- *Consensus:* All nodes calculate (in a distributed iterative manner, using only local information at each iteration) *same function* of initial values $x_1[0], x_2[0], \ldots, x_N[0]$
- *Average Consensus:* All nodes calculate (in a distributed manner) the *average* $\overline{x} \equiv \frac{1}{N}\sum_{\ell=1}^{N} x_\ell[0]$ (where N is the number of nodes)
- Possible centralized strategy: Route all values to a single entity (leading node) who then determines the function value (e.g., average) and routes it back to the nodes
- Average serves as *primitive* for estimation, inference and diagnosis (easily adjusted to arbitrary linear functions)

Outline

- Notation and mathematical preliminaries
- Average consensus in digraphs
- Ratio consensus
- Minimum-time consensus
- Examples
- Concluding remarks and future directions

Graph Notation

- *Digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
  - Nodes (system components) $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$
  - Edges (directed communication links) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ where $(v_j, v_i) \in \mathcal{E}$ iff node v_j can receive information from node v_i
  - In-neighbors $\mathcal{N}_j^- = \{v_i \mid (v_j, v_i) \in \mathcal{E}\}$; in-degree $\mathcal{D}_j^- = |\mathcal{N}_j^-|$
  - Out-neighbors $\mathcal{N}_j^+ = \{v_l \mid (v_l, v_j) \in \mathcal{E}\}$; out-degree $\mathcal{D}_j^+ = |\mathcal{N}_j^+|$
- *Adjacency matrix A:* $A(j, i) = 1$ if $(v_j, v_i) \in \mathcal{E}$; $A(j, i) = 0$ otherwise
- *Undirected graph:* $(v_j, v_i) \in \mathcal{E}$ iff $(v_i, v_j) \in \mathcal{E}$ (bidirectional links)
  In undirected graphs, we have (for each node j)
  $\mathcal{N}_j^+ = \mathcal{N}_j^-$ and $\mathcal{D}_j^+ = \mathcal{D}_j^- = \mathcal{D}_j$; also, $A = A^T$
- *(Strongly) connected (di)graph* if for any $i, j \in \mathcal{V}$, $j \neq i$, there exists a (directed) path connecting them, i.e.,

$$v_i = v_{i_0} \to v_{i_1},\ v_{i_1} \to v_{i_2},\ \ldots,\ v_{i_{t-1}} \to v_{i_t} = v_j$$

Conditions for Asymptotic Average Consensus

- Necessary and sufficient conditions on P for asymptotic average consensus [Xiao & Boyd, 2004]
  1. P has a simple eigenvalue at 1 with left eigenvector $\mathbf{1}^T = [1\ 1\ \dots\ 1]$ and right eigenvector $\mathbf{1} = [1\ 1\ \dots\ 1]^T$
  2. All other eigenvalues of P have magnitude strictly smaller than 1
- As $k \to \infty$, $P^k \to \frac{1}{N}\mathbf{1}\mathbf{1}^T$ which implies that

$$\lim_{k\to\infty} x[k] = \frac{1}{N}\mathbf{1}\mathbf{1}^T x[0] = \left(\frac{\sum_{\ell=1}^{N} x_\ell[0]}{N}\right)\mathbf{1} \equiv \overline{x}\mathbf{1}$$

- Nonnegative $p_{ji} \implies P$ is primitive bistochastic

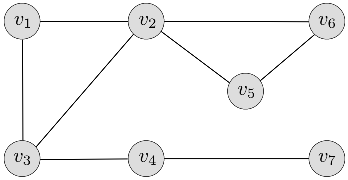
How to distributively reach the average in digraphs?

Average Consensus in Digraphs

- In undirected graphs nodes have a number of ways to distributively choose their weights so as to form a bistochastic matrix
- Digraphs not as easy to handle, even in a centralized manner (because in general $\mathcal{D}_j^+ \neq \mathcal{D}_j^-$)
- Asymptotic approaches:
  - Distributed algorithms that rely on first obtainining weights that form bistochastic matrices [Gharesifard & Cortés, 2012], [T.C. & C.N.H., 2013]
  - Distributed approaches that introduce additional state variables and use broadcast gossip [Franceschelli *et al*, 2011], [Cai & Ishii, 2011] to reach average consensus *asymptotically*
  - Run two coupled iterations simultaneously (ratio consensus) [Benezit *et al*, 2010], [A.D. Domíngez-García & C.N.H., 2010], [C.N.H. & T.C., 2011] that reach average consensus *asymptotically*

Finite-time Average Consensus in Graphs

- Finite-time approaches for *undirected* graphs:
  - *Finite-time* average consensus [J. Cortés, 2006], [S. Sundaram & C.N.H, 2007], [Wang & Xiao, 2010]
  - *Minimum-time* average consensus [Y. Yuan *et al*, 2009] (associated with final value of linear iterations)



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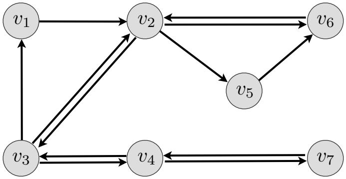

How to reach the average in *finite-time* in digraphs?

Finite-time Average Consensus in Graphs

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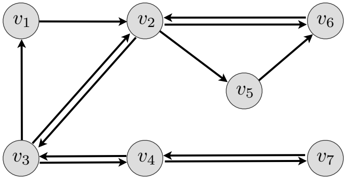

How to reach the average in *finite-time* in digraphs?

We propose an algorithm that combines *ratio consensus* with *minimum-time consensus*.

Average Consensus using Ratio Consensus
Pair of Simultaneous Linear Iterations

- Run two iterations [Benezit *et al*, 2010], [D-G & H, 2010]

$$\begin{array}{rcl|rcl} y[k+1] & = & P_c y[k] & x[k+1] & = & P_c x[k] \\ y[0] & = & [y_1[0] \ldots y_N[0]]^T & x[0] & = & \mathbf{1} \end{array}$$

- Matrix P_c st $P_c(l,j) = \frac{1}{1+\mathcal{D}_j^+}$ for $v_l \in \mathcal{N}_j^+$ (zero otherwise)
- Since P_c is primitive column stochastic, we know that as $k \to \infty$, $P_c^k \to \mathbf{v}\mathbf{1}^T$ for a *strictly positive* vector $\mathbf{v}$ such that $\mathbf{v} = P_c\mathbf{v}$ ($\mathbf{v}$ is *normalized* so that its entries sum to unity)
- This implies that

$$\begin{array}{rcl} \lim_{k\to\infty} y[k] & = & \mathbf{v}\mathbf{1}^T y[0] = \left(\sum_{\ell=1}^{N} y_\ell[0]\right)\mathbf{v} \\ \lim_{k\to\infty} x[k] & = & \mathbf{v}\mathbf{1}^T x[0] = N\mathbf{v} \end{array}$$

- For all nodes $j \in \{1, 2, \ldots, N\}$, ratio converges

$$\frac{y_j[k]}{x_j[k]} \to \frac{v_j \sum_{\ell=1}^{N} y_\ell[0]}{v_j N} = \frac{\sum_{\ell=1}^{N} y_\ell[0]}{N} \equiv \overline{y}$$

Preliminaries - I

Minimal polynomial of a matrix pair

The minimal polynomial associated with the matrix pair $[P, e_j^\top]$, denoted by $q_j(t) = t^{M_j+1} + \sum_{i=0}^{M_j} \alpha_i^{(j)} t^i$, is the monic polynomial of minimum degree $M_j + 1$ that satisfies $e_j^\top q_j(P) = 0$.

Easy to show (e.g., using the techniques in [Y. Yuan *et al*, 2009]) that

$$\sum_{i=0}^{M_j+1} \alpha_i^{(j)} w_j[k+i] = 0, \quad \forall k \in \mathbb{Z}_+ \ ,$$

where $\alpha_{M_j+1}^{(j)} = 1$. Denote z-transform of $w_j[k]$ as $W_j(z) \triangleq \mathbb{Z}(w_j[k])$. Then,

$$W_j(z) = \frac{\sum_{i=1}^{M_j+1} \alpha_i^{(j)} \sum_{\ell=0}^{i-1} w_j[\ell] z^{i-\ell}}{q_j(z)} \ ,$$

where $q_j(z)$ is the minimal polynomial of $[P, e_j^\top]$.

Preliminaries - II

Define the following polynomial:

$$p_j(z) \triangleq \frac{q_j(z)}{z-1} \triangleq \sum_{i=0}^{M_j} \beta_i^{(j)} z^i$$

The application of the final value theorem (FVT) yields:

$$\phi_y(j) = \lim_{k\to\infty} y_j[k] = \lim_{z\to 1}(z-1)Y_j(z) = \frac{\mathbf{y}_{M_j}^{\mathsf{T}}\boldsymbol{\beta}_j}{\mathbf{1}^{\mathsf{T}}\boldsymbol{\beta}_j}$$

$$\phi_x(j) = \lim_{k\to\infty} x_j[k] = \lim_{z\to 1}(z-1)X_j(z) = \frac{\mathbf{x}_{M_j}^{\mathsf{T}}\boldsymbol{\beta}_j}{\mathbf{1}^{\mathsf{T}}\boldsymbol{\beta}_j}$$

where

- $\mathbf{y}_{M_j}^{\mathsf{T}} = (y_j[0], y_j[1], \ldots, y_j[M_j])$
- $\mathbf{x}_{M_j}^{\mathsf{T}} = (x_j[0], x_j[1], \ldots, x_j[M_j])$
- $\boldsymbol{\beta}_j$ is the vector of coefficients of the polynomial $p_j(z)$

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where

- $y_{M_j}^{\mathsf{T}} = (y_j[0], y_j[1], \ldots, y_j[M_j])$
- $x_{M_j}^{\mathsf{T}} = (x_j[0], x_j[1], \ldots, x_j[M_j])$
- $\boldsymbol{\beta}_j$ is the vector of coefficients of the polynomial $p_j(z)$

How can we obtain $\boldsymbol{\beta}_j$ in the computation of final values?

Preliminaries - III

- Consider the vectors of differences between $2k+1$ successive discrete-time values of $y_j[k]$ at node v_j and $x_j[k]$:

$$\overline{y}_{2k}^{T} = (y_j[1] - y_j[0], \ldots, y_j[2k+1] - y_j[2k]),$$

$$\overline{x}_{2k}^{T} = (x_j[1] - x_j[0], \ldots, x_j[2k+1] - x_j[2k]).$$

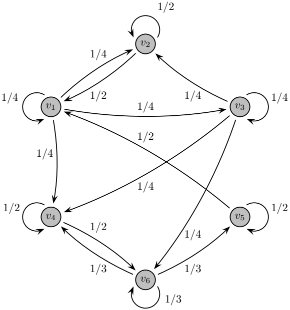
- Let us define their associated Hankel matrix:

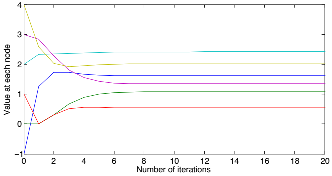
$$\Gamma\{y_{2k}^{T}\} \triangleq \begin{bmatrix} y_j[0] & y_j[1] & \ldots & y_j[k] \\ y_j[1] & y_j[2] & \ldots & y_j[k+1] \\ \vdots & \vdots & \ddots & \vdots \\ y_j[k] & y_j[k+1] & \ldots & y_j[2k] \end{bmatrix},$$

- β_j can be computed as *the kernel of the first defective Hankel matrix* for each of $\Gamma\{\overline{y}_{2k}^{T}\}$ and $\Gamma\{\overline{x}_{2k}^{T}\}$
- For arbitrary initial conditions y_0 and x_0, except a set of initial conditions with Lebesgue measure zero.

- **Input:** A strongly connected digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$
- **Data:** Successive observations for $y_j[k]$ and $x_j[k]$, $\forall v_j \in \mathcal{V}$ using simultaneous iterations for *ratio consensus* with initial conditions $y[0] = y_0$ and $x[0] = \mathbf{1}$, respectively
- **Step 1:** Each node $v_j \in \mathcal{V}$ stores the vectors of differences $\overline{y}_{M_j}^{\mathsf{T}}$ and $\overline{x}_{M_j}^{\mathsf{T}}$ between successive values of $y_j[k]$ and $x_j[k]$, respectively
- **Step 2:** Increase the dimension k of $\Gamma\{\overline{y}_{M_j}^{\mathsf{T}}\}$ and $\Gamma\{\overline{x}_{M_j}^{\mathsf{T}}\}$, until they lose rank; store their first defective matrix
- **Step 3:** The kernel $\boldsymbol{\beta}_j = \left(\beta_0, \ldots, \beta_{M_j-1}, 1\right)^{\mathsf{T}}$ of the first defective matrix gives the consensus values ϕ_y and ϕ_x
- **Step 4:** The average is computed by

$$\mu_j = \frac{\phi_y(j)}{\phi_x(j)} = \frac{y_{M_j}^{\mathsf{T}} \boldsymbol{\beta}_j}{x_{M_j}^{\mathsf{T}} \boldsymbol{\beta}_j}$$

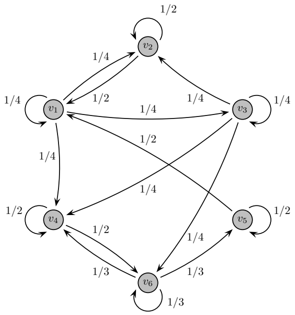


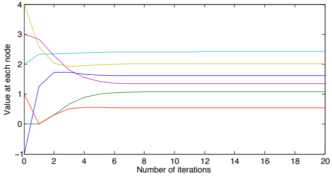


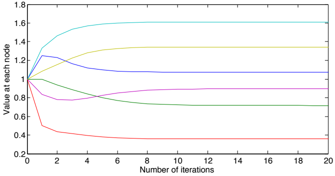
Each node can computes the exact average $\mu_j = \phi_y(j)/\phi_x(j)$, e.g., v_1

$$\mu_1 = \frac{\phi_y(1)}{\phi_x(1)} = \frac{1.6119}{1.0746} = 1.5$$

in 12 steps (see [Y. Yuan *et al*, 2009] for number of steps)







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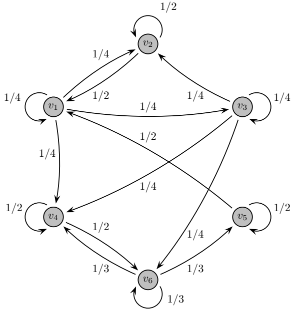

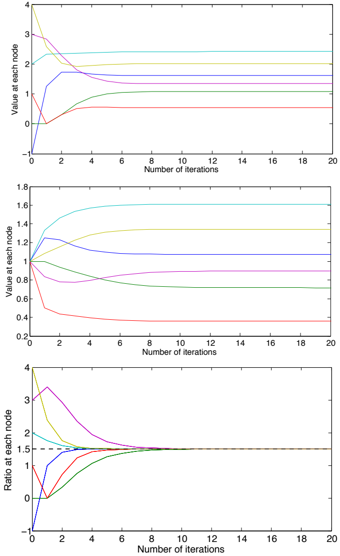

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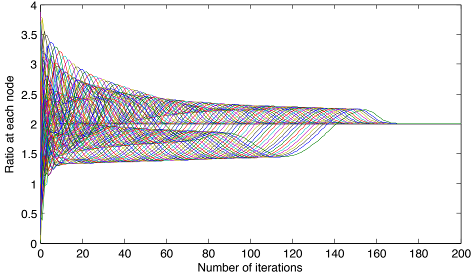
- A discrete, age-structured model of population growth
- Our example consists of a 100 nodes



- Node v_{100}, for example, computes the average in 34 steps (i.e., $2(M_{100}+1) = 2 \times 17 = 34$ steps)
- (Asymptotic) ratio-consensus algorithm seems to need more than 160 steps for the error to be small enough

Concluding Remarks and Future Directions

Conclusions:

- Proposed a distributed iterative algorithm, in which each node:
  - has knowledge of its *outgoing* links
  - reaches average consensus in *directed* graphs in *minimum-time*
  - uses only output observations at each component (finite-time history of its own values)

Future work:

- Study average consensus in a switching topology and with time-varying delays
- Consider noisy output observations

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Thank You!

Questions?

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