

# When to stop iterating in digraphs of unknown size?

## An application to finite-time average consensus

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- 1 **Distributed coordination:** the subject of extensive research for a long time
  - parallel and distributed computation, distributed optimization in sensor networks, formation control of robotic networks, dynamics of opinion forming, . . .
- 2 **Finite-time algorithms** are generally more desirable
  - they converge in finite-time
  - closed-loop systems under finite-time control usually demonstrate better disturbance rejection properties
  - important in applications where, for example, the averaging operation is a first step towards more involved control or coordination tasks

## Advertisement

Christoforos N. Hadjicostis, Alejandro D. Domínguez-García and Themistokis Charalambous (2018), *"Distributed Averaging and Balancing in Network Systems: with Applications to Coordination and Control"*, Foundations and Trends <sup>®</sup> in Systems and Control: Vol. 5: No. 2-3, pp 99–292.



# Motivation

Why is a termination mechanism needed?

- If a node computes (or estimates within an acceptable precision) its desired value, it can stop updating and communicating with other nodes
- Such an action might cause the multi-agent system to become disconnected and prevent other nodes in the system from completing their task

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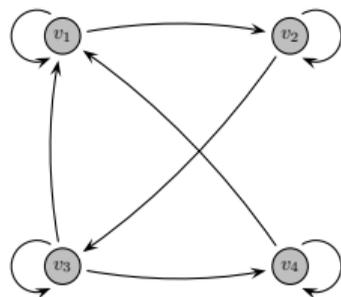
- A mechanism is needed to determine when an agreement between nodes has been reached so that nodes can terminate the distributed process

- Typically, finite-time average consensus is handled by **pre-determining the number of steps that all nodes in the network will perform**
  - ✗ requires some prior knowledge about the network and the convergence rate of the iteration process
- Finite-time **approximate** average consensus there have been two main approaches:
  - synchronous, in which nodes assume knowledge of the diameter of the network and terminate the process all together at a multiple of the diameter
    - ✗ requires global information
  - event-triggered, in which nodes terminate their process provided their value is close enough to the desired one
    - ✗ the magnitude of the error is proportional to the diameter
- Finite-time **exact** average consensus: nodes do not require any global parameter and compute the exact value
  - ✗ nodes never stop the iterations because it is impossible for them to know whether other nodes have computed the average

# Introduction

## Distributed System Model

- Distributed systems conveniently captured by digraphs
  - 1 Components represented by vertices (nodes)
  - 2 Communication and sensing links represented by edges



- Consider a network with nodes  $(v_1, v_2, \dots, v_N)$  (e.g., sensor networks, computers, etc.)
- Nodes can receive information according to (possibly directed) communication links
- Each node  $v_j$  has some initial value  $x_j[0]$  (e.g., belief, position, velocity)

# Graph Notation

- **Digraph**  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ 
  - Nodes (system components)  $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$
  - Edges (directed communication links)  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  where  $(v_j, v_i) \in \mathcal{E}$  iff node  $v_j$  can receive information from node  $v_i$
  - In-neighbors  $\mathcal{N}_j^- = \{v_i \mid (v_j, v_i) \in \mathcal{E}\}$ ; in-degree  $\mathcal{D}_j^- = |\mathcal{N}_j^-|$
  - Out-neighbors  $\mathcal{N}_j^+ = \{v_l \mid (v_l, v_j) \in \mathcal{E}\}$ ; out-degree  $\mathcal{D}_j^+ = |\mathcal{N}_j^+|$
- **Adjacency matrix**  $A$ :  $A(j, i) = 1$  if  $(v_j, v_i) \in \mathcal{E}$ ;  $A(j, i) = 0$  otherwise
- **Undirected graph**:  $(v_j, v_i) \in \mathcal{E}$  iff  $(v_i, v_j) \in \mathcal{E}$  (bidirectional links)  
In undirected graphs, we have (for each node  $j$ )  
 $\mathcal{N}_j^+ = \mathcal{N}_j^-$  and  $\mathcal{D}_j^+ = \mathcal{D}_j^- = \mathcal{D}_j$ ; also,  $A = A^T$
- **(Strongly) connected (di)graph** if for any  $i, j \in \mathcal{V}$ ,  $j \neq i$ , there exists a (directed) path connecting them, i.e.,

$$v_i = v_{i_0} \rightarrow v_{i_1}, v_{i_1} \rightarrow v_{i_2}, \dots, v_{i_{t-1}} \rightarrow v_{i_t} = v_j$$

# Preliminary results - I

## Average Consensus using Ratio Consensus

- Run two iterations [Benezit *et al*, 2010], [D-G & H, 2010]

$$\begin{array}{l|l} y[k+1] = P_c y[k] & x[k+1] = P_c x[k] \\ y[0] = [y_1[0] \dots y_N[0]]^T & x[0] = \mathbf{1} \end{array}$$

- Matrix  $P_c$  st  $P_c(l, j) = \frac{1}{1+D_j^+}$  for  $v_l \in \mathcal{N}_j^+$  (zero otherwise)
- Since  $P_c$  is primitive column stochastic, we know that as  $k \rightarrow \infty$ ,  $P_c^k \rightarrow \mathbf{v}\mathbf{1}^T$  for a *strictly positive* vector  $\mathbf{v}$  such that  $\mathbf{v} = P_c \mathbf{v}$  ( $\mathbf{v}$  is *normalized* so that its entries sum to unity)
- This implies that

$$\begin{aligned} \lim_{k \rightarrow \infty} y[k] &= \mathbf{v}\mathbf{1}^T y[0] = \left( \sum_{\ell=1}^N y_\ell[0] \right) \mathbf{v} \\ \lim_{k \rightarrow \infty} x[k] &= \mathbf{v}\mathbf{1}^T x[0] = N\mathbf{v} \end{aligned}$$

- For all nodes  $j \in \{1, 2, \dots, N\}$ , ratio converges

$$\frac{y_j[k]}{x_j[k]} \rightarrow \frac{v_j \sum_{\ell=1}^N y_\ell[0]}{v_j N} = \frac{\sum_{\ell=1}^N y_\ell[0]}{N} \equiv \bar{y}$$

# Preliminaries - II

## Minimal polynomial of a matrix pair

### Minimal polynomial of a matrix pair

The minimal polynomial associated with the matrix pair  $[P, e_j^\top]$ , denoted by

$q_j(t) = t^{M_j+1} + \sum_{i=0}^{M_j} \alpha_i^{(j)} t^i$ , is the monic polynomial of minimum degree  $M_j + 1$  that satisfies  $e_j^\top q_j(P) = 0$ .

Easy to show (e.g., using the techniques in [Y. Yuan *et al.*, 2009]) that

$$\sum_{i=0}^{M_j+1} \alpha_i^{(j)} w_j[k+i] = 0, \quad \forall k \in \mathbb{Z}_+,$$

where  $\alpha_{M_j+1}^{(j)} = 1$ . Denote  $z$ -transform of  $w_j[k]$  as  $W_j(z) \triangleq \mathbb{Z}(w_j[k])$ . Then,

$$W_j(z) = \frac{\sum_{i=1}^{M_j+1} \alpha_i^{(j)} \sum_{\ell=0}^{i-1} w_j[\ell] z^{i-\ell}}{q_j(z)},$$

where  $q_j(z)$  is the minimal polynomial of  $[P, e_j^\top]$

Define the following polynomial:

$$p_j(z) \triangleq \frac{q_j(z)}{z-1} \triangleq \sum_{i=0}^{M_j} \beta_i^{(j)} z^i$$

The application of the final value theorem (FVT) yields:

$$\phi_y(j) = \lim_{k \rightarrow \infty} y_j[k] = \lim_{z \rightarrow 1} (z-1)Y_j(z) = \frac{\mathbf{y}_{M_j}^T \boldsymbol{\beta}_j}{\mathbf{1}^T \boldsymbol{\beta}_j}$$

$$\phi_x(j) = \lim_{k \rightarrow \infty} x_j[k] = \lim_{z \rightarrow 1} (z-1)X_j(z) = \frac{\mathbf{x}_{M_j}^T \boldsymbol{\beta}_j}{\mathbf{1}^T \boldsymbol{\beta}_j}$$

where

- $\mathbf{y}_{M_j}^T = (y_j[0], y_j[1], \dots, y_j[M_j])$
- $\mathbf{x}_{M_j}^T = (x_j[0], x_j[1], \dots, x_j[M_j])$
- $\boldsymbol{\beta}_j$  is the vector of coefficients of the polynomial  $p_j(z)$

# Preliminaries - IV

How can we obtain  $\beta_j$  in the computation of final values?

- Consider the vectors of differences between  $2k + 1$  successive discrete-time values of  $y_j[k]$  and  $x_j[k]$  at node  $v_j$ :

$$\bar{y}_{2k}^T = (y_j[1] - y_j[0], \dots, y_j[2k + 1] - y_j[2k])$$

$$\bar{x}_{2k}^T = (x_j[1] - x_j[0], \dots, x_j[2k + 1] - x_j[2k])$$

- Let us define their associated Hankel matrix:

$$\Gamma\{y_{2k}^T\} \triangleq \begin{bmatrix} y_j[0] & y_j[1] & \dots & y_j[k] \\ y_j[1] & y_j[2] & \dots & y_j[k + 1] \\ \vdots & \vdots & \ddots & \vdots \\ y_j[k] & y_j[k + 1] & \dots & y_j[2k] \end{bmatrix}$$

- $\beta_j$  can be computed as **the kernel of the first defective Hankel matrix** for each of  $\Gamma\{\bar{y}_{2k}^T\}$  and  $\Gamma\{\bar{x}_{2k}^T\}$
- Works for arbitrary initial conditions  $y_0$  and  $x_0$ , except a set of initial conditions with Lebesgue measure zero

# Preliminaries - V

Computation of the average in digraphs in a finite number of steps

## Theorem [TC *et al.*, 2015])

Consider a strongly connected graph  $\mathcal{G}(\mathcal{N}, \mathcal{E})$ . Let  $y_j[k]$  and  $x_j[k]$  (for all  $v_j \in \mathcal{N}$  and  $k = 0, 1, 2, \dots$ ) be the result of the iterations for ratio consensus. Then, the solution to the average consensus can be distributively obtained in minimum number of steps at each node  $v_j$ , by computing

$$\mu_j \triangleq \lim_{k \rightarrow \infty} \frac{y_j[k]}{x_j[k]} = \frac{\phi_y(j)}{\phi_x(j)} = \frac{y_{M_j}^\top \beta_j}{x_{M_j}^\top \beta_j}.$$

- A distributed algorithm with which the exact average is computed in  $2(M_j + 1)$  time steps for each node  $v_j$

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- A distributed algorithm with which the exact average is computed in  $2(M_j + 1)$  time steps for each node  $v_j$

Since nodes do not know whether other nodes have computed the average yet, the iterations continue indefinitely

# Main results

- [Y. Yuan *et al.*, 2013] showed that after  $2(M_j + 1)$  the kernel of the Hankel matrices  $\Gamma\{\bar{y}_{2k}^T\}$  and  $\Gamma\{\bar{x}_{2k}^T\}$  for arbitrary initial conditions  $y_0$  and  $x_0$  (except a set of initial conditions with *Lebesgue measure zero*) become defective
- **Main idea:**
  - $2(M_j + 1)$  is an upper bound of the maximum distance of any other node to node  $v_j$  (which is *a priori* unknown)
  - once this distance becomes known via the finite-time consensus algorithm (Theorem 1), node  $v_j$  can use it to decide when everybody is done
  - done by observing whether the value of node  $v_j$  in a **max-consensus algorithm** has changed in  $2(M_j + 1)$  time steps
    - ✓ If it does not, then all other nodes have also computed their final value and the node can terminate the iterations
    - ✗ Otherwise, at least one other node  $v_i$  has not finished computing its average and its value participating in the max-consensus algorithm keeps increasing

# Termination algorithm

- Once ratio consensus iterations are initiated, each node  $v_j$  also initiates two counters  $c_j$ ,  $c_j[0] = 0$ , and  $r_j$ ,  $r_j[0] = 0$ 
  - Counter  $c_j$  increments by one at every time step, i.e.,  $c_j[k + 1] = c_j[k] + 1$
  - The way counter  $r_j$  updates is described next

- A max-consensus algorithm is initiated as well, given by

$$\theta_j[k + 1] = \max_{v_i \in \mathcal{N}_j \cup \{v_j\}} \{ \max\{\theta_i[k], c_i[k]\} \}, \quad \theta_j[0] = 0$$

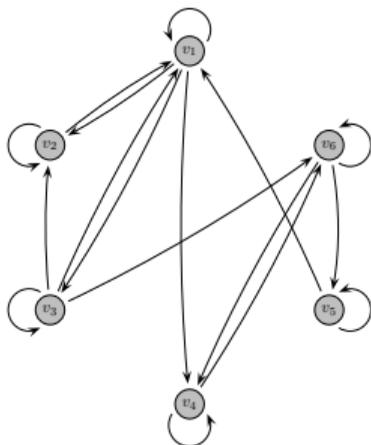
- Every time step  $k$  for which  $\theta_j[k + 1] = \theta_j[k]$ , counter  $r_j$  increments by one; otherwise,  $r_j$  is set to zero, i.e.,

$$r_j[k + 1] = \begin{cases} 0, & \text{if } \theta_j[k + 1] \neq \theta_j[k], \\ r_j[k] + 1, & \text{otherwise.} \end{cases}$$

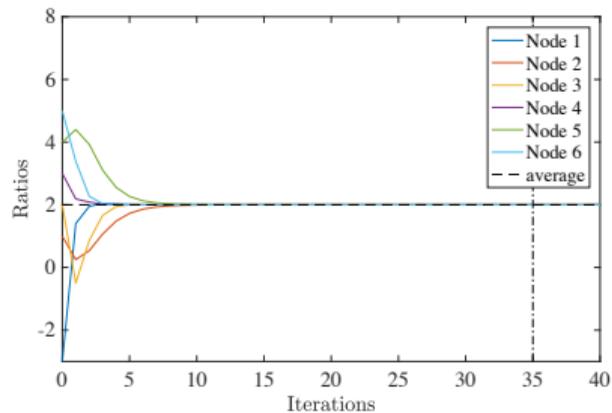
- Once the square Hankel matrices  $\Gamma\{\bar{y}_{M_j}^\top\}$  and  $\Gamma\{\bar{x}_{M_j}^\top\}$  for node  $v_j$  lose rank, node  $v_j$  saves the count of the counter  $c_j$  at that time step; note that  $c_j$  stops at  $2(M_j + 1)$
- Node  $v_j$  terminates iterations when  $r_j = 2(M_j + 1)$

# Examples

## Example 1: Simple 6-node digraph



Node number	# of steps (average)	# of steps (termination)
1	12	$18+12-1=29$
2	14	$18+14-1=31$
3	12	$18+12-1=29$
4	18	$18+18-1=35$
5	14	$18+14-1=31$
6	12	$18+12-1=29$



- The number of steps,  $T_j$ , needed for node  $v_j$  to terminate a process is

$$T_j = \max_{v_i \in \mathcal{N}} \{2(M_i + 1)\} + 2(M_j + 1) - 1$$

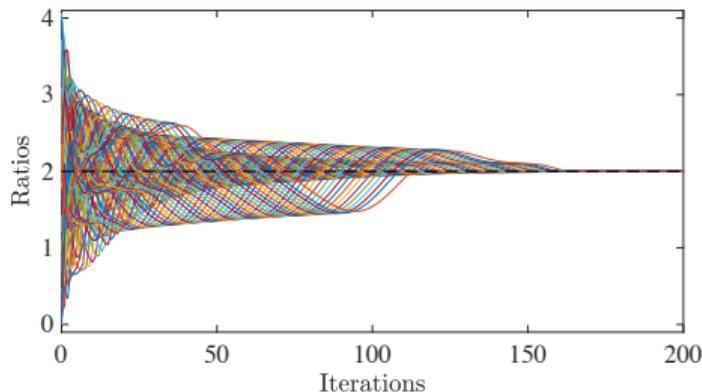
As a result,  $T_{\max}$ , is given by

$$T_{\max} = 2 \max_{v_i \in \mathcal{N}} \{2(M_i + 1)\} - 1$$

# Examples

## Example 2: Leslie network (100 nodes)

- A discrete, age-structured model of population growth
- Due to the structure of the network, the asymptotic convergence time is large:



- (Asymptotic) ratio-consensus algorithm seems to need more than 160 steps for the error to be small enough
- **By running our distributed termination algorithm, all nodes terminate their process in 75 steps**

# Thank You!



**Questions?**

**For more information:**

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## PhD Position on Wireless Networked Control Systems

- A fully-funded PhD student position is available at the Distributed Systems Control Group, Department of Electrical Engineering and Automation, School of Electrical Engineering, Aalto University
- Apply by 29th June 2018 (<http://www.aalto.fi/en/about/careers/jobs/view/1877/>)
- Available from August 2018