Robust Linear Quadratic Regulator for Uncertain Systems

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Outline

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Optimal control design for linear dynamical systems appeared in many diverse applications (e.g., aerospace, communication, robotics, finance, biology)

**Certainty Equivalence Principle**: same solution as for the deterministic problem as long as the disturbances present in the stochastic control system are zero mean

**In realistic applications** the presence of (nonzero-mean) disturbances in stochastic control systems affect optimality of the controller and compromise their performance
Introduction

Approach

- **Develop a LQR:** which is robust to disturbance variability, by using the *total variation distance* as a metric.

### Total Variation Uncertainty Ball

Set of all possible noise distributions, center at a nominal distribution $\mu$

$$
\mathbb{B}_{R_{TV}}(\mu) \triangleq \left\{ \nu_{w_i}(\cdot) \in \mathcal{M}_1([p_1, p_2]), i = 1, \ldots, N - 1 : \sum_{i=0}^{N-1} \| \nu_{w_i}(\cdot) - \mu_{w_i}(\cdot) \|_{TV} \leq R_{TV} \right\}, \quad R_{TV} \in [0, 2]
$$

- $\mathcal{M}_1([p_1, p_2])$: set of probability distributions on $[p_1, p_2]$
- $\mu(\cdot)$: “nominal” probability distribution of $w_k$
- $\nu(\cdot)$: “variation” probability distribution of $w_k$
Consider a discrete-time system

\[ x_{k+1} = A_k x_k + B_k u_k + w_k, \quad x_0 = x, \quad k = 0, \ldots, N - 1 \]

- \( x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m \): state and control vectors
- \( A_k \in \mathbb{R}^{n \times n}, B_k \in \mathbb{R}^{n \times m} \): dynamics and input matrices
- \( \{w_k : k = 0, \ldots, N - 1\} \) is independent sequence of RV's
  - unknown probability distribution \( \nu_{w_k}(dw) : k = 0, \ldots, N - 1 \)
  - zero mean and \( W_k = \mathbb{E}[w_k w_k^T] < \infty \)
Problem Formulation I

Performance Criterion

Define the N-stage expected cost

\[ J_N(\pi, \nu, x) \triangleq \mathbb{E}_x^\pi \left[ \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N \right] \]

- \( Q_k \succeq 0, R_k \succ 0 \): stage and input cost matrices
- \( \mathbb{E}_x^\pi [\cdot] \): induced by the unknown distribution \( \nu \triangleq \{ \nu_{w_k}(\cdot) \} \) of the noise sequence \( w_k \)

Minimax Stochastic Problem

\[ J_N^*(x) \triangleq \min_{u \in \mathcal{U}(x)} \max_{\nu(\cdot) \in \mathbb{B}_{RTV}(\mu)} J_N(\pi, \nu, x), \quad \forall x \in \mathcal{X} \]  \hspace{1cm} (1)

- **Minimization** is over the control laws \( u \in \mathcal{U}(x) \)
- **Maximization** is over the variation probability distribution \( \nu(\cdot) \in \mathbb{B}_{RTV}(\mu) \)
Remark

- For $R_{TV} = 0$ then (1) reduces to the standard LQR problem with a known solution\(^1\)
- The covariance of the noise $W_k$ enters in the total cost, but not the control law, i.e.,

**Control Law:** $u_k = G_k x_k$ with $G_k = -(R_k + B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k$

**Total Cost:** $J_0^*(x_0) = x_0^T P_0 x_0 + \sum_{k=0}^{N-1} \text{Tr}(P_{k+1} W_k)$

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The dynamic programming algorithm gives\(^2\)

\[
J_N^*(x_N) = x_N^T Q_N x_N \\
J_k^*(x_k) = \min_{u_k} \left\{ x_k^T Q_k x_k + u_k^T R_k u_k \\
+ \max_{\nu w_k(\cdot) \in B_{RTV}(\mu)} \mathbb{E}_{\nu w_k(\cdot)} \left[ J_{k+1}(A_k x_k + B_k u_k + w_k) \right] \right\}
\]

Define

\[
\ell_k(x_k, u_k, w_k) \triangleq J_{k+1}(A_k x_k + B_k u_k + w_k) \\
= (A_k x_k + B_k u_k + w_k)^T P_{k+1}(A_k x_k + B_k u_k + w_k) \\
+ (A_k x_k + B_k u_k + w_k)^T F_{k+1} + r_{k+1}.
\]

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Let

- $\ell_{\text{max},k}(x_k, u_k), \ell_{\text{min},k}(x_k, u_k)$: max. and min. values of (2) wrt $w_k$
- $\Sigma^o(k), \Sigma_o(k)$: corresponding support sets
- $\Sigma_j(k)$: set of indices for which (2) achieves $(j + 1)th$ smallest value
- $\ell_{\Sigma_j,k}(x_k, u_k)$ the corresponding values of the sequence in $\Sigma_j(k)$

The maximization problem is given by

$$\max_{\nu_{w_k}(\cdot) \in \mathbb{B}_{RTV}(\mu)} \mathbb{E}_{\nu_w(\cdot)} \left[ \ell_k(x_k, u_k, w_k) \right]$$

$$= \ell_{\text{max},k} \nu_{w_k}^*(\Sigma^o(k)) + \ell_{\text{min},k} \nu_{w_k}^*(\Sigma_o(k)) + \sum_{j=1}^{r} \ell_{\Sigma_j,k} \nu_{w_k}^*(\Sigma_j(k))$$

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Optimal Distribution, $\nu^* \in B_{R_{TV}}(\mu)$

The maximizing variation probability distribution of $w_k$ is given by

\[
\nu^*_{w_k}(\Sigma^o(k)) = \mu_{w_k}(\Sigma^o(k)) + \frac{\alpha}{2}
\]

\[
\nu^*_{w_k}(\Sigma_o(k)) = (\mu_{w_k}(\Sigma_o(k)) - \frac{\alpha}{2})^+
\]

\[
\nu^*_{w_k}(\Sigma_j(k)) = \left(\mu_{w_k}(\Sigma_j(k)) - \left(\frac{\alpha}{2} - \sum_{z=1}^{j} \sum_{i \in \Sigma_{z-1}(k)} \mu_{w_k}(\Sigma_i) \right)^+ \right)^+
\]

\[
\alpha = \min(R_{TV}, R_{\text{max}}), \quad R_{\text{max}} = 2(1 - \mu(\Sigma^0(k)))
\]
Equivalent Formulation

Assume that $\nu_{w_k}^*(\Sigma^o(k)) < 1$ and $\nu_{w_k}^*(\Sigma_o(k)) > 0$ and hence $\nu_{w_k}^*(\Sigma_j(k)) = \mu_{w_k}(\Sigma_j(k))$, then

$$\max_{\nu_{w_k}(\cdot) \in \mathcal{B}_{RTV}(\mu)} \mathbb{E}_{\nu_{w}(\cdot)} \left[ \ell_k(x_k, u_k, w_k) \right]$$

$$= \ell_{\max, k} \nu^*(\Sigma^o(k)) + \ell_{\min, k} \nu^*(\Sigma_o(k)) + \sum_{j=1}^{r} \ell_{\Sigma_j, k} \nu^*(\Sigma_j(k))$$

$$= \left( \ell_{\max, k} - \ell_{\min, k} \right) \frac{R_{TV}}{2} + \sum_{w_k \in \Sigma} \ell_k(w_k) \mu(w_k).$$

- The first term in the right measures the difference between the max. and min. values of $\ell_k(x_k, u_k, w_k)$ wrt $w_k$ scaled by the TV distance.
- It has the interpretation of minimizing the disturbance variability.
Solution of Minimax Problem

- By the solution of the maximization

\[ J^*_k(x_k) = \min_{u_k} \left\{ x_k^T Q_k x_k + u_k^T R_k u_k + \mathbb{E}_{v^*_w(\cdot)} \left[ \ell_k(x_k, u_k, w_k) \right] \right\} \]

- expectation performed wrt to maximizing p.d. of \( w_k \)
- vectors \( w_k \) need not have zero mean under \( v^*_w \)

- By backward induction we show that

\[ J^*_k(x_k) = x_k^T P_k x_k + x_k^T F_k + r_k \]
Solution of Robust LQR

Minimizer

The optimal control is given by

\[ u^*_k = -H_{22}^{-1}(k) \left( H_{12}^T(k)x_k + B_k^T P_{k+1} \mathbb{E} v_\nu^* \cdot [w_k] + \frac{1}{2} B_k^T F_{k+1} \right) \]

or, equivalently

\[ u^*_k = -H_{22}^{-1}(k) \left( H_{12}^T(k)x_k + R_{TV} B_k^T P_{k+1} (w^+_k - w^-_k) + \frac{1}{2} B_k^T F_{k+1} \right) \]

where

- \( w^+_k \triangleq \arg \max_{w_k \in [p_1, p_2]} J^*_{k+1}(A_k x_k + B_k u_k + w_k) \)
- \( w^-_k \triangleq \arg \min_{w_k \in [p_1, p_2]} J^*_{k+1}(A_k x_k + B_k u_k + w_k) \)
- \( H_{12}(k) \triangleq A_k^T P_{k+1} B_k \) and \( H_{22}(k) \triangleq R_k + B_k^T P_{k+1} B_k \)
- Feedback gain matrices and Riccati equations depend on the variation probability distribution of \( w_k \)
Consider the linear discrete uncertain system, with the following dynamic and input matrices

\[
A = \begin{bmatrix}
0.9974 & 0.0539 \\
-0.1078 & 1.1591
\end{bmatrix}, \quad
B = \begin{bmatrix}
0.0013 \\
0.0539
\end{bmatrix},
\]

\[
Q = \begin{bmatrix}
0.25 & 0 \\
0 & 0.05
\end{bmatrix}, \quad Q_N = Q, \quad R = 0.5
\]

- Initial conditions
  \( x_0 = [2 \ 1]^T \).
- \( w_k \) selected randomly with a known nominal probability distribution \( \mu_w \)
Optimal control and trajectories

- Left plot: standard LQR without noise
- Right plot: standard LQR with noise
Numerical Example

Robust LQR

Optimal control and trajectories

- Left plot: Robust LQR with $R_{TV} = 1$
- Right plot: Robust LQR with $R_{TV} = R_{\text{max}}$
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Conclusions

- Robust LQR with TV distance captures the disturbance variability, leading to an overall good performance.

- Control laws which are more robust wrt disturbances, at the expense of additional costs.

- Designer needs to balance the desire for low costs with undesirability of scenarios with high disturbance variability.

Possible future direction:

Extension to the case of systems with parametric uncertainties

\[ x_{k+1} = \left( A_k + \Delta A_k(w_k) \right) x_k + \left( B_k + \Delta B_k(w_k) \right) u_k \]
Thank you!

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