Infinite Horizon Average Cost DP Subject to Ambiguity on Conditional Distribution

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IEEE Conference on Decision and Control (CDC'15)
Osaka, Japan, December 2015

Outline

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Introduction - Motivation I

General Problem

Address optimality of stochastic control strategies based on infinite horizon minimax Average Cost (AC) criterion via DP subject to Total Variation (TV) distance ambiguity on conditional distribution of controlled process

Methodology:

- Formulate stochastic control problem using minimax theory
  - controller minimizes pay-off
  - conditional distribution maximizes pay-off
- Maximization of linear functionals on the space of probability meas.
- Minimax stochastic control under a Markovian assumption

Introduction - Motivation II

Infinite Horizon Markov Control Model

$$\text{MCM:}\Big(\mathcal{X},\mathcal{U},\{\mathcal{U}(x){:}x{\in}\mathcal{X}\},\{Q(dz|x,u){:}(x,u){\in}\mathcal{X}\times\mathcal{U}\},f\Big)$$

(a) $\mathcal{X}$: State Space

(b) $\mathcal{U}$: Control Space

(c) $\mathcal{U}(x)$: Feasible Controls

(d) $Q(dz|x,u)$: Controlled Process

(e) f: One-Stage-Cost

Infinite horizon Average Cost Criterion

$$J^0(g,x) \triangleq \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}_x^g\Big\{\sum_{k=0}^{n-1} f(x_k,u_k)\Big\}$$

Introduction - Motivation III

Markov Control Problem (MCP)

Find a control policy $g^* \in G$ such that

$$J^0(g^*, x) \triangleq \inf_{g \in G} J^0(g, x) = J^{0,*}(x), \quad \forall x \in \mathcal{X}$$

Under appropriate conditions:

- $\mathcal{X}, \mathcal{U}$ are of finite cardinality
- f is non-negative
- For all $g \in G_{SM}$, the controlled process distribution $Q(z|x,u)$ is an irreducible stochastic matrix

Introduction - Motivation IV

Classical DP equation[1]

There exists a pair $(J^{0,*}, V^0(x))$, which is the solution of the DP of the infinite horizon MCP

$$J^{0,*} + V^0(x) = \inf_{u \in \mathcal{U}(x)} \left\{ f(x,u) + \sum_{z \in \mathcal{X}} Q(z|x,u) V^0(z) \right\}, \quad x \in \mathcal{X}$$

Remark

- The AC optimality criterion and the DP equation depend on the complete knowledge of $Q(\cdot|x,u)$
- Any mismatch affects the optimality and robustness of the optimal decision strategies

[1] **[Kumar-Varaiya86][Lerma-Lasserre96][Bertsekas05]**

Outline

Maximization with TV Distance I

Problem Formulation

Notation:

1. (Σ, d_Σ): complete, separable metric space
2. $(\Sigma, \mathcal{B}(\Sigma))$: corresponding measurable space
3. $\mathcal{M}_1(\Sigma)$: set of probability measures on $\mathcal{B}(\Sigma)$
4. $BC^+(\Sigma)$: space of bounded continuous non-negative functions

Nominal and Class of Measures

- Nominal measure: $\mu \in \mathcal{M}_1(\Sigma)$
- Class of measures: $\nu \in \mathcal{M}_1(\Sigma)$ belongs to set

$$\mathbb{B}_R(\mu) = \{\nu \in \mathcal{M}_1(\Sigma) : ||\nu - \mu||_{TV} \leq R\}, \quad R \geq 0$$

Maximization with TV Distance II

Characterization of Maximizing Measure

Average payoff over $\mathbb{B}_R(\mu)$

$$D^+(R) = \sup_{\nu \in \mathbb{B}_R(\mu)} \mathbb{L}(\nu) \equiv \sup_{\nu \in \mathbb{B}_R(\mu)} \int_\Sigma \ell(x)\nu(dx), \quad \ell \in BC^+(\Sigma)$$

- For a given $\mu \in \mathcal{M}_1(\Sigma)$ and $\nu \in \mathbb{B}_R(\mu)$ define the set

$$\widetilde{\mathbb{B}}_R(\mu) \stackrel{\triangle}{=} \left\{ \xi \in \mathbb{M}_0(\Sigma) : \xi = \nu - \mu, \nu \in \mathcal{M}_1(\Sigma), ||\xi||_{TV} \leq R \right\}$$

Solution of Maximization

$$\begin{aligned} \int_\Sigma \ell(x)\nu(dx) &= \int_\Sigma \ell(x)\left(\xi^+(dx) - \xi^-(dx)\right) + \int_\Sigma \ell(x)\mu(dx) \\ &\leq \left\{ \sup_{x \in \Sigma} \ell(x) - \inf_{x \in \Sigma} \ell(x) \right\} \frac{||\xi||_{TV}}{2} + \int_\Sigma \ell(x)\mu(dx) \end{aligned}$$

Maximization with TV Distance III

Characterization of Maximizing Measure

- The upper bound is achieved by $\xi^* \in \widetilde{\mathbb{B}}_R(\mu)$ as follows. Let

$$x^0 \in \Sigma^0 \stackrel{\triangle}{=} \left\{ x \in \overline{\Sigma} : \ell(x) = \sup\{\ell(y) : y \in \Sigma\} \right\}$$

$$x_0 \in \Sigma_0 \stackrel{\triangle}{=} \left\{ x \in \overline{\Sigma} : \ell(x) = \inf\{\ell(y) : y \in \Sigma\} \right\}$$

- Take $\xi^*(dx) = \nu^*(dx) - \mu(dx) = \frac{R}{2}\Big(\delta_{x^0}(dx) - \delta_{x_0}(dx)\Big)$

$$\int_\Sigma \ell(x)\nu^*(dx) = \frac{R}{2}\Big\{ \sup_{x\in\Sigma} \ell(x) - \inf_{x\in\Sigma} \ell(x) \Big\} + \int_\Sigma \ell(x)\mu(dx)$$

- The first right side term is related to the oscillator seminorm of a non-negative function

Maximization with TV Distance IV

Characterization of Maximizing Measure

Extremum Pay-Off

$$\int_{\Sigma} \ell(x)\nu^*(dx) = \int_{\Sigma^0} \ell_{\max}\nu^*(dx) + \int_{\Sigma_0} \ell_{\min}\nu^*(dx) + \int_{\Sigma\setminus\Sigma^0\cup\Sigma_0} \ell(x)\mu(dx)$$

Optimal Distribution[2] $\nu^* \in \mathbb{B}_R(\mu)$

- $\int_{\Sigma^0} \nu^*(dx) = \mu(\Sigma^0) + \frac{R}{2} \in [0,1]$
- $\int_{\Sigma_0} \nu^*(dx) = \mu(\Sigma_0) - \frac{R}{2} \in [0,1]$
- $\nu^*(A) = \mu(A), \quad \forall A \subseteq \Sigma \setminus \Sigma^0 \cup \Sigma_0$

[2] **[C.D. Charalambous et al. IEEETAC-14]**

Outline

Problem Formulation I

The MiniMax Optimization Problem

Nominal Controlled Process Distribution

For each $g \in G_{SM}$,

$$Prob(x_t \in A | x^{t-1}, u^{t-1}) = Q^o(A | x_{t-1}, u_{t-1}), \quad A \in \mathcal{B}(\mathcal{X})$$

Class of Uncertain Controlled Process Distributions

Given $Q^o(\cdot|x_{t-1}, u_{t-1})$ and $R \in [0, 2]$,

$$B_R(Q^o)(x, u) \stackrel{\triangle}{=} \left\{ Q(\cdot|x, u) : ||Q(\cdot|x, u) - Q^o(\cdot|x, u)||_{TV} \leq R \right\}$$

Assumption:

- The map $f : \mathcal{X} \times \mathcal{U} \longmapsto \mathbb{R}$ is bounded continuous and non-negative

Problem Formulation II

The MiniMax Optimization Problem

Average cost per unit-time for a class

Infinite horizon AC per unit-time when policy $g \in G$ is used,

$$J(g,x) \triangleq \limsup_{n\longrightarrow\infty} \frac{1}{n} J_n(g,x)$$

$$= \limsup_{n\longrightarrow\infty} \frac{1}{n} \Big\{ \sup_{Q(\cdot|x,u)\in\mathbb{B}_R(Q^o)(x,u)} \mathbb{E}_x^g \Big[\sum_{k=0}^{n-1} f(x_k,u_k) \Big] \Big\}$$

MiniMax MCP

Find a control policy $g^* \in G$ such that

$$J(g^*,x) \triangleq \inf_{g\in G} J(g,x) = J^*(x), \quad \forall x \in \mathcal{X}$$

Problem Solution I

The General Dynamic Programming

n-stage cost

The maximization of the expected n-stage cost is given by

$$J_n(g, q_0) = \max_{Q(\cdot|x,u) \in \mathbb{B}_R(Q^o)(x,u)} q_0^T \left\{ \sum_{k=0}^{n-1} Q(g)^k \right\} f(g)$$

$$= q_0^T \left\{ \sum_{k=0}^{n-1} Q^*(g)^k \right\} f(g)$$

- $Q^*(g)$: maximizing conditional distribution

- The infinite horizon AC per unit-time is then

$$J(g, q_0) = \limsup_{n \longrightarrow \infty} \frac{1}{n} q_0^T \left\{ \sum_{k=0}^{n-1} Q^*(g)^k \right\} f(g)$$

Problem Solution II

The General Dynamic Programming

Assumption

If for all $g \in G_{SM}$, and $R \in [0, R_{\max}] \in [0, 2]$, the maximizing conditional distribution $Q^*(g)$ is an irreducible stochastic matrix, then

$$J(g, q_0) = q(g)^T f(g)$$

- $q(g)$: unique invariant distribution
- $J(g, q_0) \equiv J(g)$: independent of the initial distribution

Problem Solution III

The General Dynamic Programming

Main DP Theorem

1. There exists a solution (V, J^*) to the DP equation

$$J^* + V(x) = \min_{u \in \mathcal{U}} \left\{ f(x,u) + \sum_{z \in \mathcal{X}} Q^*(z|x,u) V(z) \right\}$$

or, equivalently, to the DP equation

$$J^* + V(x) = \min_{u \in \mathcal{U}} \left\{ f(x,u) + \sum_{z \in \mathcal{X}} Q^o(z|x,u) V(z) + \frac{R}{2} \left(\max_{z \in \mathcal{X}} V(z) - \min_{z \in \mathcal{X}} V(z) \right) \right\}$$

   - *Oscillator seminorm of the cost-to-go*

2. If $g^*(x)$ attains the minimum for every x, then g^* is an AC optimal policy, and the minimum cost is J^*

Problem Solution IV

The Policy Iteration Algorithm

Policy Iteration Algorithm

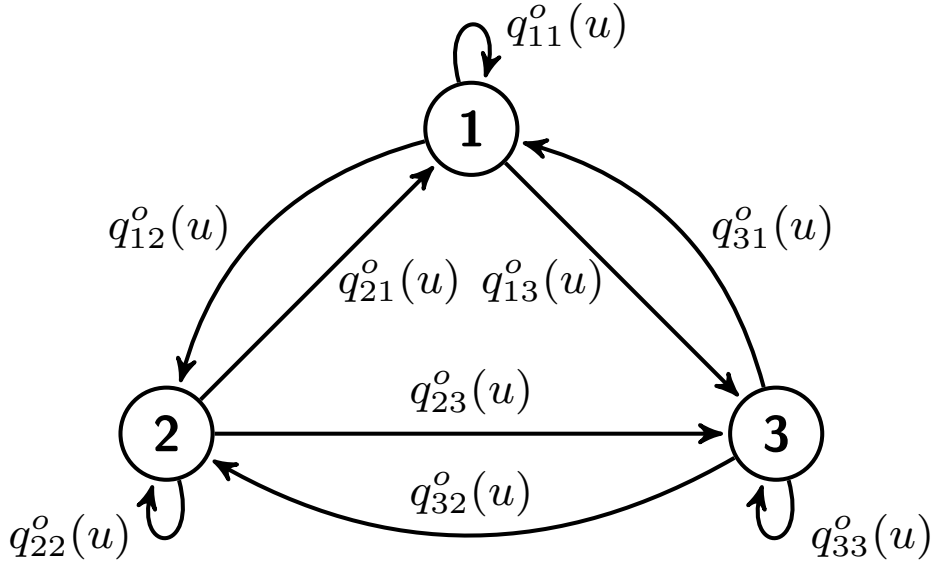
- Classical policy iteration algorithm performed using the nominal conditional distribution
- A modified algorithm for average cost DP is proposed based on TV distance ambiguity

- Policy improvement and evaluation steps performed using the maximizing conditional distribution as follows:
  a) Solve DP equation based on nominal conditional distribution
  b) Identify the support sets and calculate the maximizing distribution
  c) Solve the general DP equation

Outline

Infinite Horizon MCM Example I

- MCM: $\mathcal{X} = \{1, 2, 3\}$ and $\mathcal{U} = \{u_1, u_2\}$



Nominal transition probabilities

- Under Control u_1:

$$Q^o(u_1) = \frac{1}{9}\begin{pmatrix} 3 & 1 & 5 \\ 4 & 2 & 3 \\ 1 & 6 & 2 \end{pmatrix}$$

- Under Control u_2:

$$Q^o(u_2) = \frac{1}{9}\begin{pmatrix} 1 & 2 & 6 \\ 4 & 2 & 3 \\ 4 & 1 & 4 \end{pmatrix}$$

Infinite Horizon MCM Example II

Cost Function for each State

- Under control u_1: $f(1,u_1)=2,\ f(2,u_1)=1,\ f(3,u_1)=3$
- Under control u_2: $f(1,u_2)=.5,\ f(2,u_2)=3,\ f(3,u_2)=0$

Policy iteration algorithm ($R=6/9$)

A. Let $m=0$:

(1) Initial policies: $g_0(1)=u_1,\ g_0(2)=u_2,\ g_0(3)=u_2$

(2) **(PE)** Solve $J_{Q^o}(g_0)e+V_{Q^o}(g_0)=f(g_0)+Q^o(g_0)V_{Q^o}(g_0)$,

$$\begin{pmatrix} V_{Q^o}(g_0,1)\\ V_{Q^o}(g_0,2)\\ V_{Q^o}(g_0,3)\end{pmatrix}=\begin{pmatrix}1.8\\3.375\\0\end{pmatrix},\quad J_{Q^o}(g_0)=1.175.$$

Identify the support sets $\mathcal{X}^+=\{2\}$, $\mathcal{X}^-=\{3\}$, $\mathcal{X}_1=\{1\}$ and calculate $Q^*(u_1)$ and $Q^*(u_2)$, i.e.,

Infinite Horizon MCM Example III

$$Q^*(u_1) = \begin{pmatrix} \left(q^o_{11}(u_1) - \left(\frac{R}{2} - q^o_{13}(u_1)\right)^+\right)^+ & \min\left(1, q^o_{12}(u_1) + \frac{R}{2}\right) & \left(q^o_{13}(u_1) - \frac{R}{2}\right)^+ \\ \left(q^o_{21}(u_1) - \left(\frac{R}{2} - q^o_{23}(u_1)\right)^+\right)^+ & \min\left(1, q^o_{22}(u_1) + \frac{R}{2}\right) & \left(q^o_{23}(u_1) - \frac{R}{2}\right)^+ \\ \left(q^o_{31}(u_1) - \left(\frac{R}{2} - q^o_{33}(u_1)\right)^+\right)^+ & \min\left(1, q^o_{32}(u_1) + \frac{R}{2}\right) & \left(q^o_{33}(u_1) - \frac{R}{2}\right)^+ \end{pmatrix}$$

and similarly for $Q^*(u_2)$.

- $Q^*(u)$ remains irreducible!

Solve $J_{Q^*}(g_0)e + V_{Q^*}(g_0) = f(g_0) + Q^*(g_0)V_{Q^*}(g_0)$. The solution is

$$\begin{pmatrix} V_{Q^*}(g_0, 1) \\ V_{Q^*}(g_0, 2) \\ V_{Q^*}(g_0, 3) \end{pmatrix} = \begin{pmatrix} 1.8 \\ 3.375 \\ 0 \end{pmatrix}, \quad J_{Q^*}(g_0) = 2.3.$$

Infinite Horizon MCM Example IV

(3) (**PI**) Calculate $g_1 = \text{argmin}_{g\in\mathbb{R}^3}\{f(g) + Q^*(g)V_{Q^*}(g_0)\}$,

$$g_1(1) = u_2, \quad g_1(2) = u_1, \quad g_1(3) = u_2$$

B. Let $m = 1$: Following the same procedure as in step A we obtain

$$\begin{pmatrix} V_{Q^*}(g_1,1) \\ V_{Q^*}(g_1,2) \\ V_{Q^*}(g_1,3) \end{pmatrix} = \begin{pmatrix} 0.468 \\ 1.125 \\ 0 \end{pmatrix}, \quad J_{Q^*}(g_1) = 0.708.$$

and $g_2(1) = u_2$, $g_2(2) = u_1$, $g_2(3) = u_2$.

Since, $g_2 = g_1$, then $g^* = g_1$ is an optimal control policy with minimum cost $J_{Q^*} = 0.708$.

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Conclusions - Future Work

Conclusions

- New infinite horizon average cost DP equation, and the corresponding policy iteration algorithm

Future Work

- $(\mathcal{X},\mathcal{U})$: Borel spaces
- $R \in [R_{\max}, 2]$: irreducibility condition is violated
- Develop multichain DP equations

$$J^*(x) = \inf_{u\in\mathcal{U}(x)} \left\{ \int_{\mathcal{X}} Q^o(dz|x,u) J^*(z) + \frac{R}{2} \left(\sup_{z\in\mathcal{X}} J^*(z) - \inf_{z\in\mathcal{X}} J^*(z) \right) \right\}$$

$$J^*(x) + V(x) = \inf_{u\in\mathcal{U}(x)} \left\{ f(x,u) + \int_{\mathcal{X}} Q^o(dz|x,u) V(z) + \frac{R}{2} \left(\sup_{z\in\mathcal{X}} V(z) - \inf_{z\in\mathcal{X}} V(z) \right) \right\}$$

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