On the Rate of Convergence of Positive Linear Systems with Heterogeneous Time-Varying Delays

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July 19, 2013
ECC 13, Zurich, Switzerland
Introduction | Positive systems

Positive systems

Wide variety of applications, including

- **Social science**: population models, etc.
- **Biology/Medicine**: nitrate models, proteins, etc.,
- **Economy**: stochastic models, markov jump systems, etc.,

Why heterogeneous time delays?

Omnipresent in positive systems, in particular *distributed* systems.

Example: power control for wireless networks,

\[
\dot{p}_1(t) = -p_1(t) + a_{12}p_2(t - \tau_2^1(t)) + a_{13}p_3(t - \tau_3^1(t))
\]
Positive linear systems with heterogeneous delays

Consider the continuous-time linear system

$$
\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau^i_j(t)), \quad t \geq 0,
$$

$$
x_i(t) = \varphi_i(t), \quad t \in [-\tau_{\text{max}}, 0],
$$

where $\varphi_i(t) : [-\tau_{\text{max}}, 0] \rightarrow \mathbb{R}$ is the initial value function.

The delays are continuous functions of time, but otherwise arbitrarily varying.

We assume that $0 \leq \tau^i_j(t) \leq \tau_{\text{max}}$, and allow $\tau_{\text{max}} \rightarrow \infty$.

Definition. System is called positive if positive orthant is forward invariant:

$$
\varphi(\cdot) \geq 0 \Rightarrow x(t) \geq 0, \quad \forall t \geq 0.
$$

Fact. System is positive if and only if $A = [a_{ij}]$ is Metzler and $B = [b_{ij}]$ is non-negative [Rami 2009].
Consider the continuous-time linear system

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\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_{ij}(t)), \quad t \geq 0,
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**Definition.** System is called **positive** if positive orthant is forward invariant:

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**Positive linear systems with heterogeneous delays**

**Goal** Delay-independent stability analysis

\[
\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau^i_j(t)), \quad i = 1, \ldots, n,
\]

\[
0 \leq \tau^i_j(t) \leq \tau_{\text{max}}.
\]

**Fact.** Without delays, the positive system

\[
\dot{x}(t) = (A + B)x(t), \quad t \geq 0,
\]

is stable if and only if

\[
\begin{cases}
(A + B)^T P + P(A + B) < 0, \\
P \succ 0.
\end{cases}
\]

\[
\begin{cases}
(A + B)v < 0, \\
v > 0.
\end{cases}
\]

**Fact.** Positive systems remain *asymptotically* stable under bounded delays! [Haddad 2004, Rami 2009, Liu 2010]
Positive linear systems with heterogeneous delays

**Goal** Delay-independent stability analysis

\[
\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau_{ij}(t)), \quad i = 1, \ldots, n,
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**Fact.** Positive systems remain asymptotically stable under bounded delays!

Positive linear systems with heterogeneous delays

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Our contributions

1. Establish \textbf{exponential} stability under \textbf{heterogeneous} time-varying delays,

2. \textbf{Quantify} how \textbf{decay rate} depends on upper bound on time-varying delays,

3. \textbf{Optimizing Lyapunov function} to yield best decay-rate guarantee.
Exponential stability under bounded time-varying delays

**Theorem.** The positive linear system

\[
\begin{cases}
\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t - \tau^i_j(t)), & t \geq 0, \\
x_i(t) = \varphi_i(t), & t \in [-\tau_{\text{max}}, 0],
\end{cases}
\]

is exponentially stable if and only if there exists a vector \( v > 0 \) such that

\[(A + B)v < 0.\]

In particular, all solutions satisfy

\[\|x(t)\|_v^\infty \leq \left( \sup_{-\tau_{\text{max}} \leq s \leq 0} \|\varphi(s)\|_v^\infty \right) e^{-\eta t}, \quad t \geq 0,
\]

where \( \eta = \min_{1 \leq i \leq n} \eta_i \), and \( \eta_i \) is the unique positive solution to

\[
\left( \sum_{j=1}^{n} \frac{1}{v_i} a_{ij} v_j \right) + \left( \sum_{j=1}^{n} \frac{1}{v_i} b_{ij} v_j \right) e^{\eta_i \tau_{\text{max}}} + \eta_i = 0.
\]
Proof idea

Based on a *Lyapunov-Razumikhin* approach using

\[ V(x) = \|x\|_\infty^v = \max_{1 \leq i \leq n} \frac{x_i}{v_i}, \]

- does not impose any conditions on the evolution of the time delay,
- quantifies decay rate.
Consider linear system

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
-6 & 2 \\
1 & -3
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
3 & 0 \\
0 & 0.5
\end{bmatrix}
\begin{bmatrix}
x_1(t - \tau_1(t)) \\
x_2(t - \tau_2(t))
\end{bmatrix},
\]

\[\tau_1(t) = 5 + \sin(t),\]
\[\tau_2(t) = 4 - \cos(t).\]

- Since \(A\) is Metzler and \(B\) is non-negative, system is positive.
- If \((A + B)v < 0\) for some \(v > 0\), system is globally exponentially stable for any bounded time-varying delays:

\[
\begin{bmatrix}
-3 & 2 \\
1 & -2.5
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} < 0,
\]

\[v_1, v_2 > 0.\]

For example, \(v = (1, 1)\) verifies exponential stability.
Example

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x_1(t - \tau_1(t)) \\
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v_1 \\
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\[v_1, v_2 > 0.\]

For example, \( v = (1, 1) \) verifies exponential stability.
Example

Using vector \( v = (1, 1) \) together with \( \tau_{\text{max}} = 6 \), the solutions to the nonlinear equation

\[
\left( \sum_{j=1}^{n} \frac{1}{v_i} a_{ij} v_j \right) + \left( \sum_{j=1}^{n} \frac{1}{v_i} b_{ij} v_j \right) e^{\eta_i \tau_{\text{max}}} + \eta_i = 0, \quad i = 1, 2,
\]

are \( \eta_1 = 0.0583 \) and \( \eta_2 = 0.1957 \).
Example | Bound on decay rate

## Decay rate guarantee depends on norm

Guaranteed decay rate depends on the vector \( v > 0 \)

\[
(A + B)v < 0, \\
\left( \sum_{j=1}^{n} \frac{1}{v_i} a_{ij} v_j \right) + \left( \sum_{j=1}^{n} \frac{1}{v_i} b_{ij} v_j \right) e^{\eta_i \tau_{\text{max}}} + \eta_i = 0.
\]

Would like to maximize the smallest \( \eta_i \) under above constraints:

\[
\text{maximize} \quad \eta \\
\text{subject to} \quad \eta = \min_{1 \leq i \leq n} \eta_i, \\
(A + B)v < 0, \\
\left( \sum_{j=1}^{n} \frac{1}{v_i} a_{ij} v_j \right) + \left( \sum_{j=1}^{n} \frac{1}{v_i} b_{ij} v_j \right) e^{\eta_i \tau_{\text{max}}} + \eta_i = 0.
\]

Non-convex optimization problem in \( v \) and \( \eta_i \)!
Example | Bound on decay rate

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Would like to maximize the smallest \( \eta_i \) under above constraints:

\begin{align*}
\text{maximize} & \quad \eta \\
\text{subject to} & \quad \eta = \min_{1 \leq i \leq n} \eta_i, \\
& \quad (A + B)v < 0, \\
& \quad \left( \sum_{j=1}^{n} \frac{1}{v_i} a_{ij} v_j \right) + \left( \sum_{j=1}^{n} \frac{1}{v_i} b_{ij} v_j \right) e^{\eta_i \tau_{\text{max}}} + \eta_i = 0.
\end{align*}

Non-convex optimization problem in \( v \) and \( \eta_i \)!
Decay rate guarantee depends on norm

Guaranteed decay rate depends on the vector $v > 0$

$$(A + B)v < 0,$$

$$
\left(\sum_{j=1}^{n} \frac{1}{v_i} a_{ij} v_j \right) + \left(\sum_{j=1}^{n} \frac{1}{v_i} b_{ij} v_j \right) e^{n_i \tau_{\text{max}}} + \eta_i = 0.
$$

Would like to maximize the smallest $\eta_i$ under above constraints:

\[\text{maximize} \quad \eta \]

\[\text{subject to} \quad \eta = \min_{1 \leq i \leq n} \eta_i, \]

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$$
\left(\sum_{j=1}^{n} \frac{1}{v_i} a_{ij} v_j \right) + \left(\sum_{j=1}^{n} \frac{1}{v_i} b_{ij} v_j \right) e^{n_i \tau_{\text{max}}} + \eta_i = 0.
$$

Non-convex optimization problem in $v$ and $\eta_i$!
Optimal decay rate via convex optimization

Change-of-variables $v_i = e^{z_i}$ yields convex formulation, i.e.,

\[
\begin{align*}
\text{maximize} & \quad \eta \\
\text{subject to} & \quad \eta \leq \eta_i, \\
& \quad a_{ii} + b_{ii} + \sum_{j \neq i} (a_{ij} + b_{ij}) e^{z_j - z_i} < 0, \\
& \quad a_{ii} + \sum_{j \neq i} a_{ij} e^{z_j - z_i} + \sum_{j=1}^{n} b_{ij} e^{z_j - z_i + \eta_i \tau_{\text{max}}} + \eta_i \leq 0, \\
& \quad i = 1, \ldots, n.
\end{align*}
\]

Optimal vector $v$ and decay rate found efficiently.
Optimal decay rate via convex optimization

Change-of-variables $v_i = e^{z_i}$ yields convex formulation, i.e.,

$\textbf{maximize} \quad \eta$

$\textbf{subject to} \quad \eta \leq \eta_i,$

$$a_{ii} + b_{ii} + \sum_{j \neq i} (a_{ij} + b_{ij}) e^{z_j - z_i} < 0,$$

$$a_{ii} + \sum_{j \neq i} a_{ij} e^{z_j - z_i} + \sum_{j=1}^{n} b_{ij} e^{z_j - z_i} + \eta_i \tau_{\text{max}} + \eta_i \leq 0,$$

$i = 1, \ldots, n.$

Optimal vector $v$ and decay rate found efficiently.
Example

Simulation (black) and upper bound (blue) for two choices of $\nu$.

Left is optimal for system without delay; right is optimized for $\tau_{max} = 6$. 
Conclusions

Concluding remarks and future directions

Summary

Positive linear systems under heterogeneous time-varying delays

- Necessary and sufficient condition for exponential stability
- Best guaranteed decay rates via convex optimization

Future directions

- Exponential stability of nonlinear positive systems
- Exponential stability of positive systems with unbounded time delays
Conclusions

Concluding remarks and future directions

Summary

Positive linear systems under heterogeneous time-varying delays

- Necessary and sufficient condition for exponential stability
- Best guaranteed decay rates via convex optimization

Future directions

- Exponential stability of *nonlinear positive systems*
- Exponential stability of positive systems with *unbounded* time delays
Conclusions

Thank you!

Questions?