On the stability of the Foschini-Miljanic Algorithm with Time-Delays

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Outline

1 Introduction
   • Wireless Ad Hoc Networks
   • System Model
   • Preliminary results
   • Review of the Foschini-Miljanic Algorithm

2 Main Results
   • The Continuous-Time FM algorithm with time-delays
   • The Discrete-Time FM algorithm with time-delays
   • Illustrative Example

3 Conclusions
   • Summary
   • Future Work
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What is a wireless ad hoc network?

- A multi-hop wireless network
- A self-configuring network of nodes
- No fixed infrastructure
- Node: source, destination or relay
Network Model
- Half-duplex transceivers ⇒ Unidirectional links
- Omnidirectional antennas

Channel Model

We consider the *Physical Model* where receivers experience **interference**:

\[ I_i = \sum_{j \neq i, j \in T} g_{ji} p_j + \nu. \quad (1) \]

where

- \( g_{ij} \) the channel gain on the link between transmitter \( i \) and receiver \( j \).
- \( p_i \) the power level chosen by transmitter \( i \).
- \( \nu \) the variance of thermal noise at the receiver.
The link quality is measured by the *Signal-to-Interference-and-Noise-Ratio* (SINR). Therefore, the SINR is given by

\[
\Gamma_i = \frac{g_{ii} p_i}{\sum_{j \neq i, j \in \mathcal{T}} g_{ji} p_j + \nu}.
\]  

(2)

A transmission is successful (error free), if the SINR at the receiver is greater than the *capture ratio*, \(\gamma_i\). Therefore, we require,

\[
\frac{g_{ii} p_i}{\sum_{j \neq i, j \in \mathcal{T}} g_{ji} p_j + \nu} \geq \gamma_i
\]

(3)
Notation

\( \sigma(A) \) the spectrum of matrix \( A \)

\( \lambda(A) \) an eigenvalue of matrix \( A \)

\( \rho(A) \) the spectral radius of matrix \( A \)

\( |A| \) the element-wise absolute value of the matrix

\( A \leq B \) the element-wise inequality between \( A \) and \( B \)

\( A \geq 0 \) a nonnegative matrix

\( \det(A) \) the determinant of matrix \( A \)

\( \text{diag}(x_i) \) the matrix with elements \( x_1, x_2, \ldots \) on the leading diagonal and zeros elsewhere
When does a solution exist to such a problem?

The necessary and sufficient condition for the network to have a positive vector \( p^* \) for a positive vector \( \eta \) is that the Perron-Frobenius eigenvalue of the matrix \( C \) is less than 1, where

\[
\mathbf{p} = \begin{pmatrix} p_1 & p_2 & \ldots & p_n \end{pmatrix}^T
\]

\[
C_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \gamma_i \frac{g_{ji}}{g_{ii}}, & \text{if } i \neq j. \end{cases}
\]

\[
\eta_i = \frac{\gamma_i \nu}{g_{ii}}
\]
The Foschini-Miljanic algorithm provably succeeds in attaining the required SINRs for all nodes in the network if a solution exists and fails if there does not exist a solution.

- The continuous-time algorithm \( (k_i > 0) \)

\[
\frac{dp_i(t)}{dt} = k_i \left( -p_i(t) + \gamma_i \left( \sum_{j \neq i, j \in \mathcal{T}} g_{ji} g_{ii} p_j(t) + \frac{\nu}{g_{ii}} \right) \right)
\]  

(4)

- The discrete-time algorithm \( (k_i \in (0, 1]) \)

\[
p_i(n + 1) = (1 - k_i) p_i(n) + k_i \gamma_i \left( \sum_{j \neq i, j \in \mathcal{T}} g_{ji} g_{ii} p_j(n) + \frac{\nu}{g_{ii}} \right)
\]

(5)
Review of the Foschini-Miljanic Algorithm

Is it possible to be found in a distributed manner?

The Foschini-Miljanic algorithm provably succeeds in attaining the required SINRs for all nodes in the network if a solution exists and fails if there does not exist a solution.

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\]  

(6)

- The discrete-time algorithm \((k_i \in (0, 1])\)

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p_i(n+1) = (1 - k_i)p_i(n) + k_i \gamma_i \left( \sum_{j \neq i, j \in T} \frac{g_{ji}}{g_{ii}} p_j(n) + \frac{\nu}{g_{ii}} \right)
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- The continuous-time algorithm ($k_i > 0$)

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   - The Continuous-Time FM algorithm with time-delays
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3. **Conclusions**
   - Summary
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What happens in the presence of time-delays?

- The Foschini-Miljanic algorithms are asymptotically stable if there are no delays in the execution of the algorithms.
- Algorithms necessitate communication among users, hence propagation delays exist in the network.
- How do these algorithms behave in the presence of time-delays?
What happens in the presence of time-delays?

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What happens in the presence of time-delays?

- The Foschini-Miljanic algorithms are asymptotically stable if there are no delays in the execution of the algorithms.
- Algorithms necessitate communication among users, hence propagation delays exist in the network.
- **How do these algorithms behave in the presence of time-delays?**
The differential equation (6), when the time-delay is introduced becomes

$$\frac{dp_i(t)}{dt} = k_i \left( -p_i(t) + \gamma_i \left( \sum_{j \neq i, j \in T} \frac{g_{ji}}{g_{ii}} p_j(t - T_i) + \frac{\nu}{g_{ii}} \right) \right).$$

(8)

**Theorem 1**

If the spectral radius of matrix $C$ is less than 1, then the following power control algorithm

$$\frac{dp_i(t)}{dt} = k_i \left( -p_i(t) + \gamma_i \left( \sum_{j \neq i, j \in T} \frac{g_{ji}}{g_{ii}} p_j(t - T_i) + \frac{\nu}{g_{ii}} \right) \right), \quad i \in T$$

(9)

for $\gamma_i, g_{ji}, \nu > 0$, is asymptotically stable for arbitrarily large delays, $T_i > 0$, for any initial state $p_i(0) > 0$ and for any proportionality constant, $k_i > 0$. 
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(9)

for \( \gamma_i, \ g_{ji}, \ \nu > 0 \), is asymptotically stable for arbitrarily large delays, \( T_i > 0 \), for any initial state \( p_i(0) > 0 \) and for any proportionality constant, \( k_i > 0 \).
Sketch of the proof (1/3)

Lemma 1

Let $A$ be a nonnegative square matrix ($A \in \mathbb{R}^{N \times N}, A \geq 0$) and $B$ be a diagonal complex matrix ($B \in \mathbb{C}^{N \times N}$) whose spectral radius $\rho(B) \leq 1$, then

$$\rho(AB) \leq \rho(A).$$

- Take Laplace transform of the differential equation (9)

$$sP_i(s) - p_i(0) = -k_i \left[ P_i(s) - \gamma_i \left( \sum_{j \neq i, j \in \mathcal{T}} \frac{g_{ji}}{g_{ii}} e^{-sT_i} P_j(s) + \frac{\nu}{sg_{ii}} \right) \right]$$
Sketch of the proof (2/3)

- In matrix form this is written as

\[(I - T(s)C)P(s) = f(s)\]  \hfill (10)

where

\[T(s) = \text{diag}\left(\frac{k_i e^{-sT_i}}{s + k_i}\right)\]

\[f_i(s) = \frac{sg_{ii} p_i(0) + k_i \nu}{sg_{ii}(s + k_i)}\]

The closed-loop system is stable if \(\det(I - T(s)C)\) has no zero in the closed right-half plane. Using multivariate Nyquist criterion, it is sufficient to show that the eigenvalues of \(T(j\omega)C\) do not encircle the +1 point.
Sketch of the proof (3/3)

- It is easily shown that
  \[
  \left\{ \frac{k_i}{j\omega + k_i} e^{-j\omega T_i} : \omega, k_i, T_i \in \mathbb{R}_+ \right\} \subseteq S
  \]
  where $S$ is the unit ball, i.e. $S = \{ x : \| x \| \leq 1 \}$.

- From Lemma 1 we establish that $\rho(CT(j\omega)) \leq \rho(C) < 1$. 
Remark

Standard small-gain arguments would require $\| C \|_\infty < 1$, i.e.

\[
\frac{g_{ii}}{\sum_{j \neq i, j \in T} g_{ji}} > \gamma_i \quad \forall \ i.
\]

- $\rho(C) \leq \| C \|_\infty$, hence, this condition is more conservative.
- The return of the conservatism $\Rightarrow$ less information required.
- Way of updating the desired SINR while keeping the network functioning.
With similar arguments the Discrete Time Foschini-Miljanic algorithm now becomes

\[ p_i(n+1) = (1 - k_i)p_i(n) + k_i \gamma_i \left( \sum_{j \neq i, j \in \mathcal{I}} \frac{g_{ji}}{g_{ii}} p_j(n - n_i) + \frac{\nu}{g_{ii}} \right) \]  

(11)

where \( n_i \in \mathbb{N} \) denotes the time delays.

**Theorem 2**

If the spectral radius of matrix \( C \) is less than 1, then the discrete time algorithm

\[ p_i(n+1) = (1 - k_i)p_i(n) + k_i \gamma_i \left( \sum_{j \neq i, j \in \mathcal{I}} \frac{g_{ji}}{g_{ii}} p_j(n - n_i) + \frac{\nu}{g_{ii}} \right), \quad i \in \mathcal{I} \]  

(12)

for \( \gamma_i, g_{ji}, \nu > 0 \), is asymptotically stable for arbitrarily large delays \( (n_i \in \mathbb{N}) \) to the system, for any initial state \( p_i(0) > 0 \) and for \( k_i \) appropriately chosen \( (k_i \in (0, 1]) \).
The Discrete-Time FM algorithm with time-delays

With similar arguments the Discrete Time Foschini-Miljanic algorithm now becomes

\[ p_i(n+1) = (1 - k_i)p_i(n) + k_i \gamma_i \left( \sum_{j \neq i, j \in \mathcal{T}} \frac{g_{ji}}{g_{ii}} p_j(n-n_j) + \frac{\nu}{g_{ii}} \right) \]  

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Sketch of the proof (1/2)

- Taking z-Transforms to the discrete time algorithm (12), we have

\[ zP_i(z) - p_i(0) = (1 - k_i)P_i(z) + k_i \gamma_i \left( \sum_{j \neq i, j \in T} \frac{g_{ji}}{g_{ii}} P_j(z)z^{-n_i} + \frac{\nu z}{g_{ii}(z-1)} \right) \]

- In matrix form this is written as

\[
(z - 1)F(z)P(z) = f(z) \tag{13}
\]

where

\[ f_i(z) = k_i \gamma_i \frac{z \nu}{g_{ii}} + (z - 1)p_i(0), \]

\[
F_{ij}(z) = \begin{cases} 
  z - 1 + k_i, & \text{if } i = j, \\
  -k_i \gamma_i \frac{g_{ji}}{g_{ii}} z^{-n_i}, & \text{if } i \neq j.
\end{cases}
\]
Sketch of the proof (2/2)

- $F(z)$ can be written as $F(z) = (z - 1)I + K - KD(z)C$ where $D(z) = \text{diag}(z^{-n_i})$.
- The stability condition is equivalent to the following: the eigenvalues of

$$
\tilde{D}(e^{j\theta})C = [(e^{j\theta} - 1)I + K]^{-1}KD(e^{j\theta})C
$$

should not encircle the point $+1$, as $\theta$ varies from $0$ to $2\pi$.
- Sufficient to find the conditions for $k_i$ for which

$$
\left| \frac{k_i}{e^{j\theta} - 1 + k_i} \right| \leq 1 \forall i \in \mathcal{T}.
$$

- From Lemma 1 we establish $\rho(C\tilde{D}(e^{j\theta})) \leq \rho(C) < 1$
Sketch of the proof (2/2)

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- Sufficient to find the conditions for $k_i$ for which

$$\left| \frac{k_i}{e^{j\theta} - 1 + k_i} \right| \leq 1 \ \forall \ i \in T.$$ 

- From Lemma 1 we establish $\rho(C\tilde{D}(e^{j\theta})) \leq \rho(C) < 1$.
**Proposition 1 - Refinement on the proportionality constant** $k_i$

If the system is asymptotically stable (i.e. $\rho(C) < 1$), then the discrete-time FM algorithm is *locally* asymptotically stable for

$$k_i \in \left(0, \frac{2}{1 + \rho(C)}\right).$$

Useful whenever there is a centralized controller/base station that has knowledge of the network and is able to disseminate this information to the users.
Illustrative Example (1/3)

For this example we have that $\gamma_i = 3$ and $v = 0.04$ Watts. The initial power $p_i(0)$ for each transmitter is 1 Watt.

$$C = \begin{pmatrix} 0 & 0.5405 & 0.3880 & 0.1131 \\ 0.2143 & 0 & 0.0101 & 0.0323 \\ 0.0522 & 0.0070 & 0 & 0.0271 \\ 0.0084 & 0.0016 & 0.0385 & 0 \end{pmatrix}$$

Remark – Small-gain Argument

$\rho(C) = 0.3759$ but $\|C\|_\infty > 1$. Hence, require smaller SINR, such that

$$\sum_j C(i,j) < 1,$$

i.e., $\gamma_1 < 2.8802.$
Illustrative Example (2/3)

Figure: Discrete-time FM algorithm with delays ($T = \{15, 2, 17, 14\}$). The algorithm asymptotically converges to the desired SINR in a distributed manner.
Illustrative Example (3/3)

Figure: Proportionality constant $k = 1.4$. The system converges to the desired SINR and to the minimum power vector.

The maximum proportionality constant for which the system is locally asymptotically stable is given by

$$k < \frac{2}{1 + \rho(C)} = \frac{2}{1.3659} = 1.4643.$$
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Focused on power adaptation in an environment where time-delays exist between communicating pairs.

Introduced delays to both the continuous-time and discrete-time Foschini-Miljanic algorithms.

Proved global asymptotic stability for both algorithms in the presence of arbitrarily large constant delays.

Refined the upper bound of the proportionality constant of the discrete-time FM algorithm.
Future Work

- Extend our results to cover the time-varying delay case.
- Investigate how the convergence rate of the algorithms changes with time-delays.
Lemma 1

Let $A$ be a nonnegative square matrix ($A \in \mathbb{R}^{N \times N}, A \geq 0$) and $B$ be a diagonal complex matrix ($B \in \mathbb{C}^{N \times N}$) whose spectral radius $\rho(B) \leq 1$, then $\rho(AB) \leq \rho(A)$.

As background for the proof of this lemma, we need the following result:

Theorem

Let $A \in \mathbb{C}^{N \times N}$ and $B \in \mathbb{R}^{N \times N}$, with $B \geq 0$. If $|A| \leq B$, then

$$\rho(A) \leq \rho(|A|) \leq \rho(B).$$
Note that $AB \leq |AB| \leq |A||B|$. Since $A$ is a nonnegative matrix with real entries, then $|A| = A$. In addition $|B| \leq I$, where $I$ is the identity matrix of appropriate dimensions. Therefore, $|A||B| \leq A$. Thus, from Theorem 1,

$$\rho(AB) \leq \rho(|AB|) \leq \rho(|A||B|) \leq \rho(A).$$
Proposition 1

If the system is asymptotically stable (i.e. $\rho(C) < 1$), then the discrete-time FM algorithm is globally asymptotically stable for

$$k_i \in \left(0, \frac{2}{1 + \rho(C)}\right).$$

Since $|D(e^{j\theta})| = I$, the stability condition is equivalent to establishing that the spectral radius of

$$\tilde{C}(e^{j\theta}) = C[(e^{j\theta} - 1)I + K]^{-1}K$$
Proof of Proposition 1 (2/2)

Since \([e^{j\theta} - 1]I + K\)^{-1}K is a diagonal matrix, by Lemma 1, we can equivalently find the conditions on \(k_i\) for which

\[
\left| \frac{k_i \rho(C)}{e^{j\theta} - 1 + k_i} \right| < 1 \quad \forall \ i \in \mathcal{T}.
\]

Equivalent to the following inequality

\[
k_i^2 (\rho^2 - 1) + 2k_i - 2 < 2(k_i - 1) \cos \theta.
\]

Therefore,

\[
k_i \in \left(0, \frac{2}{1 + \rho(C)}\right).
\]