

# Average Consensus in the Presence of Dynamically Changing Directed Topologies and Time Delays

**Themistoklis Charalambous\*** and Christoforos N. Hadjicostis<sup>#</sup>

\* Department of Automatic Control, Royal Institute of Technology (KTH)

<sup>#</sup>Department of Electrical and Computer Engineering, University of Cyprus



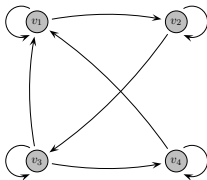
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- Motivation - Introduction
- Notation and mathematical preliminaries
- Average consensus in digraphs
  - Ratio consensus
  - Ratio consensus with time-delays
- **Contribution:** Ratio consensus with dynamically changing directed topologies and time-delays
- Examples
- Concluding remarks and future directions

# Introduction

## Distributed System Model

- Distributed systems conveniently captured by digraphs
  - 1 Components represented by vertices (nodes)
  - 2 Communication and sensing links represented by edges



- Consider a network with nodes  $(v_1, v_2, \dots, v_N)$   
(e.g., sensor networks, network of mappers in MapReduce, etc.)
- Nodes can receive information according to (possibly directed) communication links
- Each node  $v_j$  has some **initial value  $x_j[0]$**   
(e.g., measurement, position, velocity, etc.)

# Consensus and Average Consensus

- **Typical objective:** Calculate functions of initial values in a distributed manner (e.g.,  $\max_{\ell} \{x_{\ell}[0]\}$ ,  $\sum_{\ell} x_{\ell}^2[0]$ , etc.)
- **Consensus:** All nodes calculate (in a distributed manner, each time using only local information) *same function* of initial values  $x_1[0]$ ,  $x_2[0]$ ,  $\dots$ ,  $x_N[0]$
- **Average Consensus:** All nodes calculate (in a distributed manner) the *average*  $\bar{x} \equiv \frac{1}{N} \sum_{\ell=1}^N x_{\ell}[0]$  (where  $N$  is the number of nodes)
- Possible centralized strategy: Route all values to a single entity (leading node) who then determines the function value (e.g., average) and routes it back to the nodes
- Average serves as primitive for estimation, inference and diagnosis (easily adjusted to arbitrary linear functions)

# Graph Notation

- **Digraph**  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ 
  - Nodes (system components)  $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$
  - Edges (directed communication links)  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  where  $(v_j, v_i) \in \mathcal{E}$  iff node  $v_j$  can receive information from node  $v_i$
  - In-neighbors  $\mathcal{N}_j^- = \{v_i \mid (v_j, v_i) \in \mathcal{E}\}$ ; in-degree  $\mathcal{D}_j^- = |\mathcal{N}_j^-|$
  - Out-neighbors  $\mathcal{N}_j^+ = \{v_i \mid (v_i, v_j) \in \mathcal{E}\}$ ; out-degree  $\mathcal{D}_j^+ = |\mathcal{N}_j^+|$
- **Adjacency matrix**  $A$ :  $A(j, i) = 1$  if  $(v_j, v_i) \in \mathcal{E}$ ;  $A(j, i) = 0$  otherwise
- **Undirected graph**:  $(v_j, v_i) \in \mathcal{E}$  iff  $(v_i, v_j) \in \mathcal{E}$  (bidirectional links)  
In undirected graphs, we have (for each node  $j$ )  
 $\mathcal{N}_j^+ = \mathcal{N}_j^-$  and  $\mathcal{D}_j^+ = \mathcal{D}_j^- = \mathcal{D}_j$ ; also,  $A = A^T$
- **(Strongly) connected (di)graph** if for any  $i, j \in \mathcal{V}, j \neq i$ , there exists a (directed) path connecting them, i.e.,

$$v_i = v_{i_0} \rightarrow v_{i_1}, v_{i_1} \rightarrow v_{i_2}, \dots, v_{i_{t-1}} \rightarrow v_{i_t} = v_j$$

# Conditions for Reaching Average Consensus

## Linear Iterations

- **Average Consensus:** All nodes calculate (in a distributed manner) the average  $\frac{1}{N} \sum_{\ell=1}^N x_{\ell}[0]$  of initial values
- Consider linear iterations of the form

$$x_j[k+1] = p_{jj}x_j[k] + \sum_{i \in \mathcal{N}_j^-} p_{ji}x_i[k], \quad \forall j \in \{1, 2, \dots, N\}$$

where  $p_{ji}$  are constant weights and  $x_j[0]$  is the initial value of node  $j$

- Can be written in compact form as

$$\begin{aligned} x[k+1] &= Px[k] \\ x[0] &= [x_1[0] \ x_2[0] \ \dots \ x_N[0]]^T \end{aligned}$$

- Weight matrix  $P$  such that  $P(j, i) = p_{ji}$   
**Note:**  $P(j, i) = 0$  if  $(j, i) \notin \mathcal{E}$

# Conditions for Asymptotic Average Consensus

- Necessary and sufficient conditions on  $P$  for asymptotic average consensus [Xiao & Boyd, 2004]
  - 1  $P$  has a simple eigenvalue at 1 with left eigenvector  $\mathbf{1}^T = [1 \ 1 \ \dots \ 1]$  and right eigenvector  $\mathbf{1} = [1 \ 1 \ \dots \ 1]^T$
  - 2 All other eigenvalues of  $P$  have magnitude strictly smaller than 1
- As  $k \rightarrow \infty$ ,  $P^k \rightarrow \frac{1}{N} \mathbf{1} \mathbf{1}^T$  which implies that

$$\lim_{k \rightarrow \infty} x[k] = \frac{1}{N} \mathbf{1} \mathbf{1}^T x[0] = \left( \frac{\sum_{\ell=1}^N x_{\ell}[0]}{N} \right) \mathbf{1} \equiv \bar{x} \mathbf{1}$$

- Nonnegative  $p_{ji} \implies P$  is primitive bistochastic

How to distributively reach the average in digraphs?

# Average Consensus in Digraphs

- In undirected graphs nodes have a number of ways to distributively choose their weights so as to form a bistochastic matrix
- Digraphs not as easy to handle, even in a centralized manner (because in general  $\mathcal{D}_j^+ \neq \mathcal{D}_j^-$ )
- Approaches:
  - Distributed algorithms to obtain weights that form bistochastic matrices [Gharesifard & Cortés, 2012], [Rikos *et al*, 2014]
  - Distributed approaches that introduce additional state variables and use broadcast gossip [Franceschelli *et al*, 2011], [Cai & Ishii, 2011] to reach average consensus *asymptotically*
  - Run two coupled iterations simultaneously (ratio consensus) [Benezit *et al*, 2010], [A.D. Domínguez-García & C.N.H., 2010] that reach the average consensus *asymptotically*



# Average Consensus using Ratio Consensus

## Pair of Simultaneous Linear Iterations

- Run two iterations [Benezit *et al*, 2010], [A.D. Domínguez-García & C.N.H., 2010]

$$\begin{array}{l|l} \pi[k+1] = P_c \pi[k] & x[k+1] = P_c x[k] \\ \pi[0] = [\pi_1[0] \dots \pi_N[0]]^T & x[0] = \mathbf{1} \end{array}$$

- Matrix  $P_c$  st  $P_c(i, j) = \frac{1}{1+D_j^+}$  for  $v_i \in \mathcal{N}_j^+$  (zero otherwise)
- Since  $P_c$  is primitive column stochastic, we know that as  $k \rightarrow \infty$ ,  $P_c^k \rightarrow \mathbf{v}\mathbf{1}^T$  for a *strictly positive* vector  $\mathbf{v}$  such that  $\mathbf{v} = P_c \mathbf{v}$  ( $\mathbf{v}$  is *normalized* so that its entries sum to unity)
- This implies that

$$\begin{aligned} \lim_{k \rightarrow \infty} \pi[k] &= \mathbf{v}\mathbf{1}^T \pi[0] = \left( \sum_{\ell=1}^N \pi_\ell[0] \right) \mathbf{v} \\ \lim_{k \rightarrow \infty} x[k] &= \mathbf{v}\mathbf{1}^T x[0] = N\mathbf{v} \end{aligned}$$

- For all nodes  $j \in \{1, 2, \dots, N\}$ , ratio converges

$$\frac{\pi_j[k]}{x_j[k]} \rightarrow \frac{v_j \sum_{\ell=1}^N \pi_\ell[0]}{v_j N} = \frac{\sum_{\ell=1}^N \pi_\ell[0]}{N} \equiv \bar{\pi}$$

# Average Consensus in the Presence of Delays

- A transmission on link  $(v_j, v_i)$  (from node  $v_i$  to node  $v_j$ ) at time-step  $k$  can be delayed by  $0, 1, \dots, D$  steps (arriving at node  $v_j$  respectively at time-step  $k + 1, k + 2, \dots, k + D + 1$ )
- **Unknown** time-dependent delay function for each link  $(v_j, v_i)$

$$\text{delay}_{(j,i)}[k] = d, \quad d \in \{0, 1, 2, \dots, D\}$$

- $\text{delay}_{(j,j)}[k] = 0$  for all  $k$  and  $v_j \in \mathcal{V}$   
(own value is always available)
- For convenience, we use indicator functions

$$I_{k,j,i}(d) = \begin{cases} 1, & \text{if } \text{delay}_{(j,i)}[k] = d \\ 0, & \text{otherwise} \end{cases}$$

**Can ratio consensus reach average consensus in the presence of  
time-varying delays?**

# Average Consensus in the Presence of Delays

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**Can ratio consensus reach average consensus in the presence of time-varying delays?**

- Ratio consensus can be adopted to work in the presence of **bounded time-varying delays** [C.N.H. & T.C., 2011], [C.N.H. & T.C., 2014]

# Average Consensus in the Presence of Delays

## Running Two Delayed Iterations

- Run two delayed iterations [C.N.H. & T.C., 2011], [C.N.H. & T.C., 2014]

$$\begin{aligned}\pi_j[k+1] &= \rho_{jj}\pi_j[k] + \sum_{i \in \mathcal{N}_j^-} \rho_{ji} \sum_{d=0}^D \pi_i[k-d] l_{k-d,j,i}(d) \\ \pi[0] &= [\pi_1[0] \dots \pi_N[0]]^T\end{aligned}$$

$$\begin{aligned}x_j[k+1] &= \rho_{jj}x_j[k] + \sum_{i \in \mathcal{N}_j^-} \rho_{ji} \sum_{d=0}^D x_i[k-d] l_{k-d,j,i}(d) \\ x[0] &= \mathbf{1}\end{aligned}$$

- Transmissions from node  $v_i$  to node  $v_j$  at time step  $k$  undergo **identical** delays in both iterations (e.g., the pair of values are put in the same packet)
- The ratio  $\pi_j[k]/x_j[k]$  converges in the limit to  $\frac{\sum_{\ell=1}^N \pi_\ell[0]}{N}$  for all nodes  $v_j \in \mathcal{V}$

**Can ratio consensus also handle **time-varying topologies** on top of delays?**

# Average Consensus in the Presence of Delays and Switchings

- **Assumptions:**

- ① At each time  $k$ , each node  $v_j$  knows the number of nodes receiving its message - this assumption can be relaxed
- ② The delays are bounded
- ③ There exist paths between any pair of nodes infinitely often

- Run two delayed iterations **with time-varying coefficients**

$$\begin{aligned}\pi_j[k+1] &= p_{jj}[k]\pi_j[k] + \sum_{d=0}^D \sum_{v_i \in \mathcal{N}_j^-[k-d]} \pi_{j \leftarrow i}[k-d] l_{k-d,ji}[d] \\ \pi[0] &= [\pi_1[0] \dots \pi_N[0]]^T\end{aligned}$$

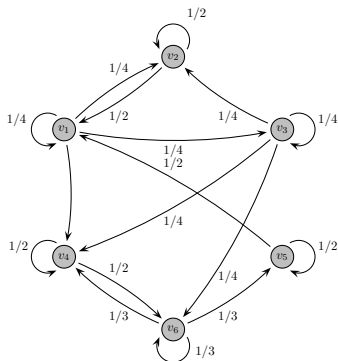
$$\begin{aligned}x_j[k+1] &= p_{jj}[k]x_j[k] + \sum_{d=0}^D \sum_{v_i \in \mathcal{N}_j^-[k-d]} x_{j \leftarrow i}[k-d] l_{k-d,ji}[d] \\ x[0] &= \mathbf{1}\end{aligned}$$

where  $x_{j \leftarrow i}[k-d] \triangleq p_{ji}[k-d]x_i[k-d]$  is the value sent from node  $v_i$  to node  $v_j$  at time step  $k-d$  that suffers delay  $d$

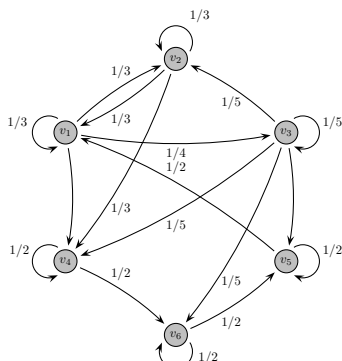
# Examples

Example: 6-node digraph (1/2)

- Initial conditions  $\pi[0] = (-1 \ 1 \ 2 \ 3 \ 4 \ 3)^T$  and  $x[0] = \mathbf{1}$
- Each node  $v_j$  chooses its self-weight and the weight of the links to its out-neighbors to be  $(1 + \mathcal{D}_j^+[k])^{-1}$



Connections and weights at time-instant  $k_1$

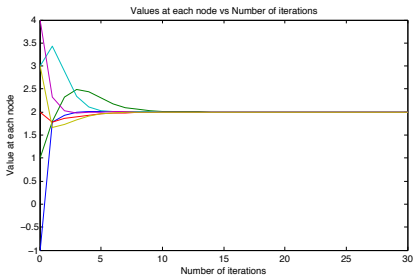


Connections and weights at time-instant  $k_2$

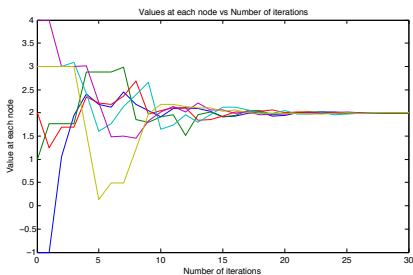
# Examples

## Example: 6-node digraph (2/2)

- Iterations converge to the average
- maximum delay  $\bar{\tau} = 5$



Switchings, no delays



With switchings and delays

# Sketch of proof (1/2)

- To handle delays (see [C.N.H. & T.C., 2014]) we introduce  $DN$  “virtual” nodes (for a total of  $(D + 1)N$  nodes) so that we can write

$$\bar{x}[k + 1] = P[k]\bar{x}[k],$$

where

$$P[k] \triangleq \begin{pmatrix} P_0[k] & I_{N \times N} & 0 & \cdots & 0 \\ P_1[k] & 0 & I_{N \times N} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{D-1}[k] & 0 & 0 & \cdots & I_{N \times N} \\ P_D[k] & 0 & 0 & \cdots & 0 \end{pmatrix},$$

with

$$\bar{x}[k] = \left( x^T[k] \ x^{(1)}[k] \ \dots \ x^{(D)}[k] \right)^T, \quad x^{(d)}[k] = \left( x_1^{(d)}[k] \ \dots \ x_N^{(d)}[k] \right)$$

- There exists  $\ell$  and an infinite sequence of time instants  $t_0, t_1, \dots, t_m, \dots$ , such that for any  $m \in \mathbb{Z}_+$ ,  $0 < t_{m+1} - t_m \leq \ell < \infty$ ,  $m \in \mathbb{Z}_+$ , and the union of graphs  $\mathcal{G}[t_m], \mathcal{G}[t_m + 1], \dots, \mathcal{G}[t_{m+1} - 1]$  is strongly connected (assumption 3)



# Sketch of proof (2/2)

- Let  $\bar{P}_{t_{m+1}-t_m} \triangleq P[t_{m+1} - 1]P[t_{m+1} - 2] \dots P[t_m]$
- The union of graphs  $\mathcal{G}[t_m], \mathcal{G}[t_m + 1], \dots, \mathcal{G}[t_{m+1} - 1]$  is strongly connected and each matrix involved in the product has strictly positive elements on the diagonal  $\Rightarrow \bar{P}_{t_{m+1}-t_m}$  is **Stochastic, Indecomposable and Aperiodic (SIA)**
- Products of matrices of the form  $\bar{P}_{t_{m+1}-t_m}$  are SIA
- **Wolfowitz theorem:** For any  $\epsilon > 0$ , there exists a finite integer  $\nu(\epsilon) \in \mathbb{N}$ , such that a finite word  $W$  given by the product of a collection of  $\nu$  stochastic matrices of the form  $\bar{P}_{t_{m+1}-t_m}$  has all of its columns approximately the same
- This implies that

$$\begin{aligned}\lim_{k \rightarrow \infty} \pi[k] &= \mathbf{v}[k] \mathbf{1}^T \pi[0] = \left( \sum_{\ell=1}^N \pi_{\ell}[0] \right) \mathbf{v}[k] \\ \lim_{k \rightarrow \infty} x[k] &= \mathbf{v}[k] \mathbf{1}^T x[0] = N \mathbf{v}[k]\end{aligned}$$

- Therefore,

$$\lim_{k \rightarrow \infty} \frac{\pi_j[k]}{x_j[k]} = \frac{\left( \sum_{\ell=1}^N \pi_{\ell}[0] \right) \mathbf{v}_j[k]}{N \mathbf{v}_j[k]} = \frac{\sum_{\ell=1}^N \pi_{\ell}[0]}{N}$$





## Conclusions:

- Proposed a distributed algorithm
  - operates **in directed networks**
  - has limited **local information exchanges**
  - converges to the average in the presence of delays and switchings






## Future work:

- Investigation of quantized average consensus in directed graphs with switching topologies and delays

# References

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**Questions?**

**For more information:**

[themisc@kth.se](mailto:themisc@kth.se)

[chadjic@ucy.ac.cy](mailto:chadjic@ucy.ac.cy)

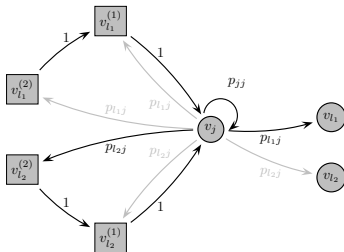
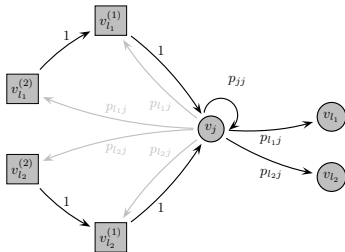


**Joker slides**

# What if a node leave the network?

By an example (1/2)

- Max number of time-slots required to receive acknowledgement is 2
- Node  $v_j$  directs the weighted messages of the links that no longer exist as **delayed information to itself**

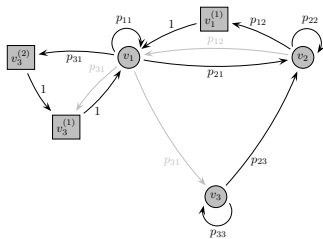


# What if a node leave the network?

By an example (2/2)

## Acknowledgement can be:

- via a **distress signal**
- via a path that reaches  $v_j$



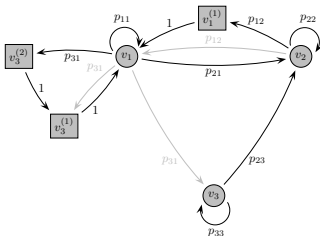


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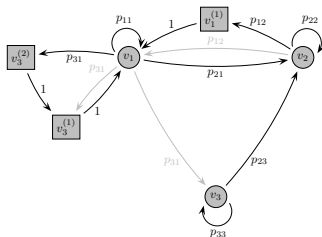
What if communication is not officially terminated?

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By an example (2/2)

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What if communication is not officially terminated?

- Each node  $v_j$  broadcasts its own state, as well as the sum of all the values, called the *total mass* in [Vaidya et al, 2012]