Robust Linear Quadratic Regulator for Uncertain Systems

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Outline

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Introduction

Motivation

- Optimal control design for linear dynamical systems appeared in many diverse applications
(e.g., aerospace, communication, robotics, finance, biology)

- **Certainty Equivalence Principle:** same solution as for the deterministic problem as long as the disturbances present in the stochastic control system are zero mean

- **In realistic applications** the presence of (nonzero-mean) disturbances in stochastic control systems affect optimality of the controller and compromise their performance

Introduction I

Approach

- **Develop a LQR:** , which is robust to disturbance variability, by using the *total variation distance* as a metric

Total Variation Uncertainty Ball

Set of all possible noise distributions, center at a nominal distribution μ

$$\mathbb{B}_{R_{TV}}(\mu) \triangleq \Big\{ \nu_{w_i}(\cdot) \in \mathscr{M}_1([p_1,p_2]), i=1,\ldots,N-1 : \sum_{i=0}^{N-1} ||\nu_{w_i}(\cdot) - \mu_{w_i}(\cdot)||_{TV} \leq R_{TV} \Big\}, \quad R_{TV} \in [0,2]$$

- $\mathscr{M}_1([p_1,p_2])$: set of probability distributions on $[p_1,p_2]$
- $\mu(\cdot)$: "nominal" probability distribution of w_k
- $\nu(\cdot)$: "variation" probability distribution of w_k

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Problem Formulation I

Linear System

Consider a discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \quad x_0 = x, \quad k = 0, \dots, N-1$$

- $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$: state and control vectors
- $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times m}$: dynamics and input matrices
- $\{w_k : k = 0, \dots, N-1\}$ is independent sequence of RV's
  - unknown probability distribution $\{\nu_{w_k}(dw) : k = 0, \dots, N-1\}$
  - zero mean and $W_k = \mathbb{E}[w_k w_k^T] < \infty$

Problem Formulation I

Performance Criterion

Define the N-stage expected cost

$$J_N(\pi, \nu, x) \triangleq \mathbb{E}_x^{\pi}\Big[\sum_{k=0}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N\Big]$$

- $Q_k \succeq 0$, $R_k \succ 0$: stage and input cost matrices
- $\mathbb{E}_x^{\pi}[\cdot]$: induced by the unknown distribution $\nu \triangleq \{\nu_{w_k}(\cdot)\}$ of the noise sequence w_k

Minimax Stochastic Problem

$$J_N^*(x) \triangleq \min_{u \in \mathscr{U}(x)} \max_{\nu(\cdot) \in \mathbb{B}_{R_{TV}}(\mu)} J_N(\pi, \nu, x), \quad \forall x \in \mathscr{X} \tag{1}$$

- **Minimization** is over the control laws $u \in \mathscr{U}(x)$
- **Maximization** is over the variation probability distribution $\nu(\cdot) \in \mathbb{B}_{R_{TV}}(\mu)$

Problem Formulation II

Performance Criterion

Remark

- For $R_{TV} = 0$ then (1) reduces to the standard LQR problem with a known solution[1]
- The covariance of the noise W_k enters in the total cost, but not the control law, i.e.,

Control Law: $u_k = G_k x_k$ with $G_k = -(R_k + B_k^T P_{k+1} B_k)^{-1} B_k^T P_{k+1} A_k$

Total Cost: $J_0^*(x_0) = x_0^T P_0 x_0 + \sum_{k=0}^{N-1} \mathrm{Tr}(P_{k+1} W_k)$

[1] B.D.O. Anderson and J.B. Moore, *Optimal Control: Linear Quadratic Methods*, Upper Saddle River, NJ, USA:Prentice-Hall, Inc., 1990.

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Solution of Robust LQR

Dynamic Programming

The dynamic programming algorithm gives[2]

$$
\begin{aligned}
J_N^*(x_N) &= x_N^T Q_N x_N \\
J_k^*(x_k) &= \min_{u_k} \Big\{ x_k^T Q_k x_k + u_k^T R_k u_k \\
&\qquad + \max_{\nu_{w_k}(\cdot) \in \mathbb{B}_{R_{TV}}(\mu)} \mathbb{E}_{\nu_{w_k}(\cdot)} \Big[J_{k+1}(A_k x_k + B_k u_k + w_k) \Big] \Big\}
\end{aligned}
$$

Define

$$
\begin{aligned}
\ell_k(x_k, u_k, w_k) &\triangleq J_{k+1}(A_k x_k + B_k u_k + w_k) \\
&= (A_k x_k + B_k u_k + w_k)^T P_{k+1} (A_k x_k + B_k u_k + w_k) \\
&\quad + (A_k x_k + B_k u_k + w_k)^T F_{k+1} + r_{k+1}. \qquad (2)
\end{aligned}
$$

[2] I.Tzortzis, C.D. Charalambous and T.Charalambous, *Dynamic Programming Subject to Total Variation Distance Ambiguity*, SICON, Volume 53(4), pp.2040–2075, 2015.

Solution of Robust LQR

Solution of Maximization (I)

Let

- $\ell_{\max,k}(x_k, u_k)$, $\ell_{\min,k}(x_k, u_k)$: max. and min. values of (2) wrt w_k
- $\Sigma^o(k)$, $\Sigma_o(k)$: corresponding support sets
- $\Sigma_j(k)$: set of indices for which (2) achieves $(j+1)th$ smallest value
- $\ell_{\Sigma_j,k}(x_k, u_k)$ the corresponding values of the sequence in $\Sigma_j(k)$

The maximization problem is given by[3]

$$\max_{\nu_{w_k}(\cdot)\in\mathbb{B}_{R_{TV}}(\mu)} \mathbb{E}_{\nu_w(\cdot)}\Big[\ell_k(x_k, u_k, w_k)\Big]$$

$$= \ell_{\max,k}\nu^*_{w_k}(\Sigma^o(k)) + \ell_{\min,k}\nu^*_{w_k}(\Sigma_o(k)) + \sum_{j=1}^{r} \ell_{\Sigma_j,k}\nu^*_{w_k}(\Sigma_j(k))$$

[3] C.D. Charalambous, I.Tzortzis, S.Loyka and T.Charalambous, *Extremum Problems with Total Variation Distance and their Applications*, IEEETAC, Volume 59(9), pp.2353–2368, 2014.

Solution of Robust LQR

Solution of Maximization (II)

Optimal Distribution, $\nu^* \in \mathbb{B}_{R_{TV}}(\mu)$

The maximizing variation probability distribution of w_k is given by

$$\nu^*_{w_k}(\Sigma^o(k)) = \mu_{w_k}(\Sigma^o(k)) + \frac{\alpha}{2}$$

$$\nu^*_{w_k}(\Sigma_o(k)) = (\mu_{w_k}(\Sigma_o(k)) - \frac{\alpha}{2})^+$$

$$\nu^*_{w_k}(\Sigma_j(k)) = \Big(\mu_{w_k}(\Sigma_j(k)) - \Big(\frac{\alpha}{2} - \sum_{z=1}^{j} \sum_{i \in \Sigma_{z-1}(k)} \mu_{w_k}(\Sigma_i)\Big)^+\Big)^+$$

$$\alpha = \min(R_{TV}, R_{\max}), \quad R_{\max} = 2(1 - \mu(\Sigma^0(k)))$$

Solution of Robust LQR

Solution of Maximization (III)

Equivalent Formulation

Assume that $\nu^*_{w_k}(\Sigma^o(k)) < 1$ and $\nu^*_{w_k}(\Sigma_o(k)) > 0$ and hence $\nu^*_{w_k}(\Sigma_j(k)) = \mu_{w_k}(\Sigma_j(k))$, then

$$\max_{\nu_{w_k}(\cdot)\in\mathbb{B}_{R_{TV}}(\mu)} \mathbb{E}_{\nu_w(\cdot)}\Big[\ell_k(x_k,u_k,w_k)\Big]$$

$$= \ell_{\max,k}\nu^*(\Sigma^o(k)) + \ell_{\min,k}\nu^*(\Sigma_o(k)) + \sum_{j=1}^{r} \ell_{\Sigma_j,k}\nu^*(\Sigma_j(k))$$

$$= \Big(\ell_{\max,k} - \ell_{\min,k}\Big)\frac{R_{TV}}{2} + \sum_{w_k\in\Sigma} \ell_k(w_k)\mu(w_k).$$

- The first term in the right measures the difference between the max. and min. values of $\ell_k(x_k,u_k,w_k)$ wrt w_k scaled by the TV distance
- It has the interpretation of minimizing the disturbance variability

Solution of Robust LQR

Solution of Minimax Problem

Solution of Minimax Problem

- By the solution of the maximization

$$J_k^*(x_k) = \min_{u_k} \left\{ x_k^T Q_k x_k + u_k^T R_k u_k + \mathbb{E}_{\nu_{w_k}^*(\cdot)} \left[\ell_k(x_k, u_k, w_k) \right] \right\}$$

  - expectation performed wrt to maximizing p.d. of w_k
  - vectors w_k need not have zero mean under $\nu_{w_k}^*$

- By backward induction we show that

$$J_k^*(x_k) = x_k^T P_k x_k + x_k^T F_k + r_k$$

Solution of Robust LQR

Minimizer

The optimal control is given by

$$u_k^* = -H_{22}^{-1}(k)\Big(H_{12}^T(k)x_k + B_k^T P_{k+1}\mathbb{E}_{\nu^*_{w_k}(\cdot)}[w_k] + \frac{1}{2}B_k^T F_{k+1}\Big)$$

or, equivalently

$$u_k^* = -H_{22}^{-1}(k)\Big(H_{12}^T(k)x_k + R_{TV} B_k^T P_{k+1}(w_k^+ - w_k^-) + \frac{1}{2}B_k^T F_{k+1}\Big)$$

where

- $w_k^+ \triangleq \arg\max_{w_k \in [p_1, p_2]} J_{k+1}^*(A_k x_k + B_k u_k + w_k)$
- $w_k^- \triangleq \arg\min_{w_k \in [p_1, p_2]} J_{k+1}^*(A_k x_k + B_k u_k + w_k)$
- $H_{12}(k) \triangleq A_k^T P_{k+1} B_k$ and $H_{22}(k) \triangleq R_k + B_k^T P_{k+1} B_k$
- Feedback gain matrices and Riccatti equations depend on the variation probability distribution of w_k

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Numerical Example

System description

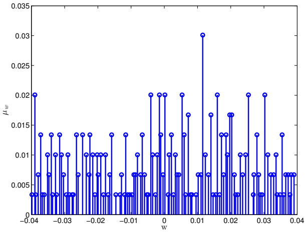
Example

Consider the linear discrete uncertain system, with the following dynamic and input matrices

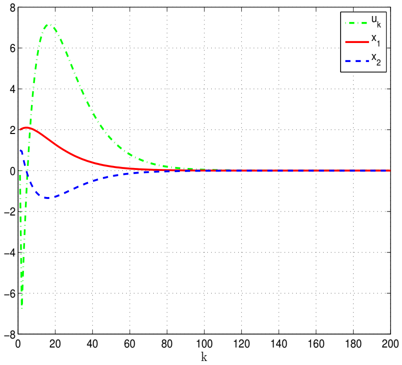
$$A = \begin{bmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0013 \\ 0.0539 \end{bmatrix},$$

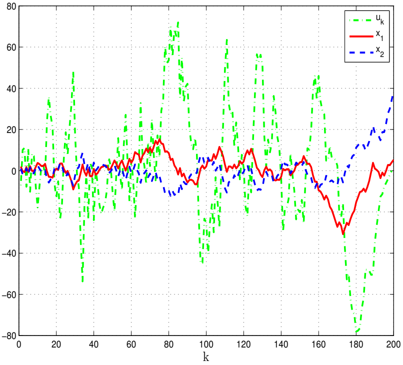
$$Q = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad Q_N = Q, \quad R = 0.5$$

- Initial conditions $x_0 = [2 \quad 1]^T$.
- w_k selected randomly with a known nominal probability distribution μ_w



Numerical Example
Standard LQR



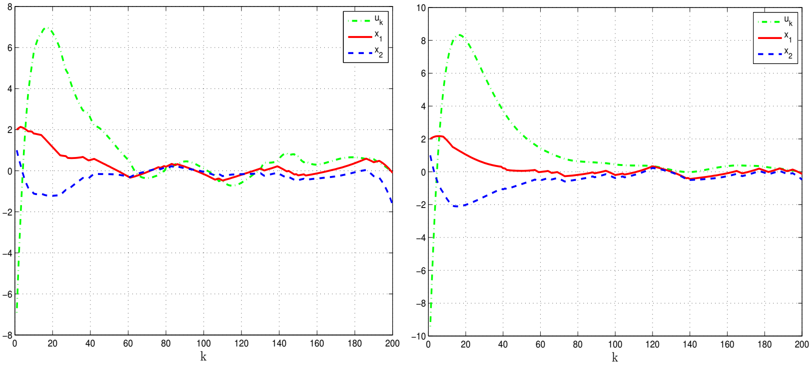


Optimal control and trajectories

- Left plot: standard LQR without noise
- Right plot: standard LQR with noise

Numerical Example

Robust LQR



Optimal control and trajectories

- Left plot: Robust LQR with $R_{TV} = 1$
- Right plot: Robust LQR with $R_{TV} = R_{\max}$

Outline

Conclusions - Future Work

Conclusions

- Robust LQR with TV distance captures the disturbance variability, leading to an overall good performance
- Control laws which are more robust wrt disturbances, at the expense of additional costs
- Designer needs to balance the desire for low costs with undesirability of scenarios with high disturbance variability

Possible future direction:

Extension to the case of systems with parametric uncertainties

$$x_{k+1} = \Big(A_k + \Delta A_k(w_k)\Big)x_k + \Big(B_k + \Delta B_k(w_k)\Big)u_k$$

Thank you!

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