Delay-independent Stability of Cone-invariant Monotone Systems

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Monotone systems

The system

$$\begin{cases} \dot{x}(t) = f\big(x(t)\big), & t \geq 0, \\ x(0) = x_0 \end{cases}$$

is called **monotone** if

$$x_0 \leq x_0' \Longrightarrow x(t, x_0) \leq x(t, x_0'), \quad \forall t \geq 0$$

Wide variety of applications, including

- molecular biology
- chemical reaction networks
- power control in wireless networks

Monotone systems with delays

Consider the continuous-time nonlinear system

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t-\tau(t))\big), \quad & t \geq 0, \\ x(t) &= \varphi(t), & t \in [-\tau_{\max}, 0], \end{aligned}$$

where $\varphi(t) : [-\tau_{\max}, 0] \to \mathbb{R}$ is the initial condition.

Fact. System is monotone if f is *cooperative*, i.e.,

$$\frac{\partial f_i}{\partial x_j}(x) \geq 0, \quad i \neq j, \, \forall x \in \mathbb{R}_+^n,$$

and g is *order-preserving*, i.e.,

$$x \leq y \;\Rightarrow\; g(x) \leq g(y)$$

Definition. Monotone system is called **positive** if

$$\varphi(t) \geq 0 \;\Rightarrow\; x(t) \geq 0, \quad \forall t \geq 0.$$

Monotone systems with delays

Delays may render an otherwise stable system unstable.

Fact. Positive monotone systems remain asymptotically stable under *constant* time delays!

Does this insensitivity property hold also for

- monotone systems which are **not** positive?
- **time-varying** delays?

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Our contributions

- Establish delay-independent stability conditions for **cone-invariant** monotone systems with **time-varying** delays

- Prove insensitivity of **global** asymptotic stability of sub-homogeneous monotone systems to time-varying delays

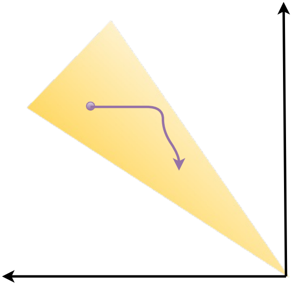
Cone-invariant monotone systems

Definition. The monotone system

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t-\tau(t))\big), \quad && t \geq 0, \\ x(t) &= \varphi(t), && t \in [-\tau_{\max}, 0], \end{aligned}$$

is called **cone-invariant** with respect to a cone $\mathcal{K}$ if

$$\varphi(t) \in \mathcal{K} \;\Rightarrow\; x(t, \varphi) \in \mathcal{K}, \quad \forall t \geq 0$$



Asymptotic stability under time-varying delays

Theorem

If there exists a vector $v \in \textbf{int}\, \mathcal{K}$ such that

$$f(v) + g(v) \in -\textbf{int}\, \mathcal{K},$$

then the cone-invariant monotone system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t - \tau(t))\big), \quad t \geq 0,$$

is asymptotically stable for any initial conditions satisfying

$$\varphi(t) \leq v, \quad t \in [-\tau_{\max}, 0]$$

When $\mathcal{K} = \mathbb{R}^n_+$, the delay-independent stability condition becomes

$$\begin{cases} f(v) + g(v) & < 0, \\ \qquad v & > 0 \end{cases}$$

Example

Consider nonlinear system

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-2x_1(t) + \frac{x_2(t)}{x_2(t)+2}\\ -2x_2(t) + \frac{x_1(t)}{x_1(t)+2}\end{bmatrix} + \begin{bmatrix}x_1(t-\tau(t))\\ x_2(t-\tau(t))\end{bmatrix},$$

$$\tau(t) = 4 + \sin(t).$$

- f is cooperative and g is order-preserving. Thus, the system is monotone.
- Since

$$\begin{cases} f(3,3)+g(3,3) & < (0,0), \\ (3,3) & > (0,0) \end{cases}$$

the vector $v = (3,3)$ verifies asymptotic stability for initial conditions

$$\varphi(t) \leq v, \quad t \in [-\tau_{\max}, 0].$$

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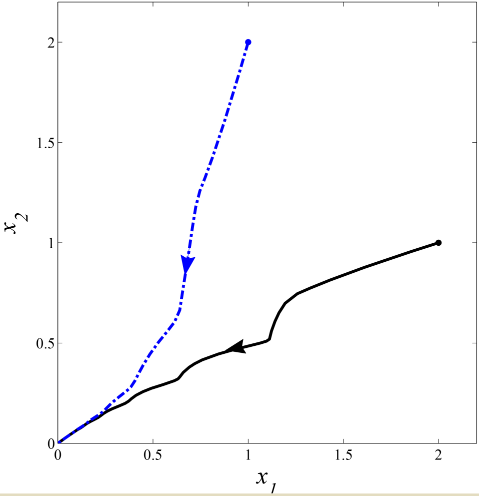

Sub-homogeneous systems

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called **sub-homogeneous** of degree $\alpha > 0$ if

$$f(\lambda x) \leq \lambda^\alpha f(x), \quad \forall x \in \mathbb{R}^n,\ \forall \lambda \geq 1$$

- includes linear mappings $f(x) = Ax$ and homogeneous vector fields

$$f(\lambda x) = \lambda^\alpha f(x), \quad \forall x \in \mathbb{R}^n,\ \forall \lambda > 0$$

When f and g are sub-homogeneous, the monotone system

$$\begin{aligned} \dot{x}_i(t) &= f_i\big(x(t)\big) + g_i\big(x_1(t-\tau_1^i(t)), \ldots, x_n(t-\tau_n^i(t))\big), && t \geq 0, \\ x_i(t) &= \varphi_i(t), && t \in [-\tau_{\max}, 0], \end{aligned}$$

is called sub-homogeneous monotone.

Insensitivity of sub-homogeneous monotone systems

Theorem. The sub-homogeneous monotone system

$$\begin{aligned} \dot{x}_i(t) &= f_i\big(x(t)\big) + g_i\big(x_1(t-\tau_1^i(t)), \ldots, x_n(t-\tau_n^i(t))\big), \quad && t \geq 0, \\ x_i(t) &= \varphi_i(t), && t \in [-\tau_{\max}, 0] \end{aligned}$$

is globally asymptotically stable if

$$\dot{x}_i(t) = f_i\big(x(t)\big) + g_i\big(x(t)\big), \quad t \geq 0$$

is globally asymptotically stable.

Proof sketch

Step 1: If the sub-homogeneous monotone system

$$\dot{x}_i\big(t\big) = f_i\big(x(t)\big) + g_i\big(x(t)\big), \quad t \geq 0$$

is globally asymptotically stable, then there exists $v \in \textbf{int}\,\mathcal{K}$ such that

$$f(v) + g(v) \in -\textbf{int}\,\mathcal{K}$$

Step 2: According to our first result,

$$\dot{x}_i\big(t\big) = f_i\big(x(t)\big) + g_i\big(x_1(t - \tau_1^i(t)), \ldots, x_n(t - \tau_n^i(t))\big), \quad t \geq 0$$

is also stable.

Concluding remarks

Summary

Cone-invariant monotone systems with time-varying delays

- Delay-independent stability condition for asymptotic stability
- Delay-insensitivity of sub-homogeneous monotone systems
- Analogous results for discrete-time monotone systems

Future directions

- Stability of cone-invariant monotone systems with *unbounded* time delays
- Delay-insensitivity of more general classes of monotone systems

Thank you!

Questions?