Delay-independent stability of sub-homogeneous cone-invariant monotone systems

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***Abstract*— Recent results in the literature have shown that particular classes of positive monotone systems are insensitive to time-varying delays, giving the impression that the delay-insensitivity property stems from the fact that the system is positive. Nonetheless, it has been lately shown that a linear cone-invariant system is insensitive to time-varying delays, asserting that the property of delay-independence may stem from the fact that the system is cone-invariant rather than positive. In this paper, we attest this claim by showing that a significant class of cone-invariant monotone systems, which includes linear cone-invariant systems as a special case, is globally asymptotically stable for any bounded time-varying delays if the corresponding delay-free system is globally asymptotically stable.**

I. Introduction

Roughly speaking, a system is called monotone if a partial order between initial states is preserved by the system dynamics (see, for example, [1]–[3]). Positive systems constitute a special case, where the nonnegative orthant (which forms a cone) is invariant. Dynamical systems with invariant sets that are cones are encountered in a wide range of application areas, including biochemistry [4]–[6], epidemiology [7], power systems [8], multi-agent systems [9], [10] and wireless networks [11]–[13].

Physical systems are usually modeled based on the assumption that their evolution depends only on the current values of the state variables. However, in many cases, the system state may also be affected by previous values of the states. For example, delays are inherent in distributed systems due to communication and processing delays, forcing subsystems to act and update their internal states based on delayed information. For this reason, the study of stability and control of dynamical systems with delayed states is important and has attracted a lot of interest. It is well known that time delays limit the performance of closed-loop control systems and may even render an otherwise stable system unstable [14]. Recent results (e.g., [15]–[19]) have revealed that a general class of nonlinear positive systems, which includes positive linear systems as a special case, is insensitive to bounded time-varying delays. In other words, a delayed system in that class is asymptotically stable if the corresponding delay-free system is asymptotically stable.

Recent work by Tanaka *et al.* [20] and Shen and Zheng [21] show that the stability of linear cone-invariant delay differential systems is insensitive to bounded time-varying delays. These results expand the class of systems that are insensitive to delays and lead to the conjecture that the insensitivity to time delays is due to cone-preservation, for which nonnegativity serves as a special case.

In this work, we provide additional evidence for this conjecture, by establishing delay-independent stability of continuous-time sub-homogeneous and discrete-time sub-homogeneous (of degree smaller than or equal to one) cone-invariant monotone systems. These systems include linear cone-invariant systems as a special case and are, to the best of the authors' knowledge, the first results that prove delay insensitivity for nonlinear conve-invariant systems.

The rest of the paper is organized as follows. In Section II the notation and preliminaries needed for the development of our results are presented. The main results are presented in Sections III and IV for discrete- and continuous-time monotone systems, respectively. Illustrative examples are given during the presentation of the results. Finally, conclusions and future directions are given in Section V.

II. Notation and Preliminaries

A. Notation

Vectors are written in bold lower case letters and matrices in capital letters. We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of natural numbers and the set of natural numbers including zero, respectively. The non-negative orthant of the n-dimensional real space $\mathbb{R}^n$ is represented by $\mathbb{R}^n_+$. For a real interval $[a, b]$ and an open set $\mathcal{W} \subseteq \mathbb{R}^n$, $\mathcal{C}\big([a, b], \mathcal{W}\big)$ denotes the space of all real-valued continuous functions on $[a, b]$ taking values in $\mathcal{W}$. For a set $\mathcal{K}$ and a matrix $A \in \mathbb{R}^{m\times n}$, by $A\mathcal{K}$ we mean that $A\mathcal{K} = \{A\boldsymbol{x} : \boldsymbol{x} \in \mathcal{K}\}$. The upper-right Dini-derivative of a continuous function $h : \mathbb{R} \to \mathbb{R}$ is denoted by $D^+h(\cdot)$.

B. Preliminaries

Next, we review the key definitions and results necessary for developing the main results of this paper. We start with the definition of a *cone*:

Definition 1 *A set $\mathcal{K} \subseteq \mathbb{R}^n$ is called a cone if, for every $\boldsymbol{x} \in \mathcal{K}$ and $\theta \in \mathbb{R}_+$, we have $\theta\boldsymbol{x} \in \mathcal{K}$.*

A cone $\mathcal{K} \subseteq \mathbb{R}^n$ is called *solid* if its interior, denoted by **int** $\mathcal{K}$, is nonempty, and it is called *pointed* if $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

Definition 2 *A cone $\mathcal{K} \subseteq \mathbb{R}^n$ is called a proper cone, if it is convex, closed, solid, and pointed.*

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A proper cone $\mathcal{K} \subseteq \mathbb{R}^n$ induces partial orderings $\leq_\mathcal{K}$ and $<_\mathcal{K}$ on $\mathbb{R}^n$ as follows:

$$\boldsymbol{x} \leq_\mathcal{K} \boldsymbol{y} \Longleftrightarrow \boldsymbol{y} - \boldsymbol{x} \in \mathcal{K},$$
$$\boldsymbol{x} <_\mathcal{K} \boldsymbol{y} \Longleftrightarrow \boldsymbol{y} - \boldsymbol{x} \in \textbf{int}\,\mathcal{K}.$$

Note that when $\mathcal{K} = \mathbb{R}^n_+$, the partial orderings $\leq_\mathcal{K}$ and $<_\mathcal{K}$ are the usual ordering $\leq$ and $<$ on $\mathbb{R}^n$, respectively.

Definition 3 *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a cone. The set*

$$\mathcal{K}^\star = \left\{ \boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{x}^\top \boldsymbol{y} \geq 0 \text{ for all } \boldsymbol{x} \in \mathcal{K} \right\}$$

is called the dual cone of $\mathcal{K}$.

The following definition introduces *cross-positive* and $\mathcal{K}$-*positive* matrices.

Definition 4 *Let $A \in \mathbb{R}^{n\times n}$ and $\mathcal{K} \subseteq \mathbb{R}^n$ be a cone. The square matrix A is said to be cross-positive on $\mathcal{K}$ if for any $\boldsymbol{x} \in \mathcal{K}$ and any $\boldsymbol{y} \in \mathcal{K}^\star$ with $\boldsymbol{y}^\top \boldsymbol{x} = 0$, we have $\boldsymbol{y}^\top A\boldsymbol{x} \geq 0$. It is called $\mathcal{K}$-positive if $A\mathcal{K} \subseteq \mathcal{K}$.*

We now define *cooperative* vector fields.

Definition 5 *A vector field $\boldsymbol{f} : \mathcal{K} \to \mathbb{R}^n$ which is continuously differentiable on the convex cone $\mathcal{K} \subseteq \mathbb{R}^n$ is said to be cooperative with respect to $\mathcal{K}$ if the Jacobian matrix $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{a})$ is cross-positive on $\mathcal{K}$ for all $\boldsymbol{a} \in \mathcal{K}$.*

The following definition introduces *sub-homogeneous vector fields*.

Definition 6 *A vector field $\boldsymbol{f} : \mathcal{K} \to \mathbb{R}^n$ is said to be sub-homogeneous of degree $\alpha > 0$ with respect to $\mathcal{K}$ if*

$$\boldsymbol{f}(\lambda \boldsymbol{x}) \leq_\mathcal{K} \lambda^\alpha \boldsymbol{f}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathcal{K},\ \forall \lambda \geq 1.$$

Finally, we define *order-preserving vector fields*.

Definition 7 *A vector field $\boldsymbol{g} : \mathcal{K} \to \mathbb{R}^n$ is called order-preserving on $\mathcal{K}$ if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$ such that $\boldsymbol{x} \leq_\mathcal{K} \boldsymbol{y}$, it holds that $\boldsymbol{g}(\boldsymbol{x}) \leq_\mathcal{K} \boldsymbol{g}(\boldsymbol{y})$.*

III. Discrete-Time Monotone Systems

A. Problem Statement

Consider the discrete-time nonlinear dynamical system

$$\Sigma : \begin{cases} \boldsymbol{x}(t+1) &= \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{g}\big(\boldsymbol{x}(t-\tau(t))\big), t \in \mathbb{N}_0, \\ \boldsymbol{x}(t) &= \boldsymbol{\varphi}(t), \qquad\qquad t \in \{-\tau_{\max}, \ldots, 0\}, \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state variable, $f, g : \mathcal{K} \to \mathbb{R}^n$ are continuous vector fields on the proper cone $\mathcal{K} \subseteq \mathbb{R}^n$, $\boldsymbol{\varphi} : \{-\tau_{\max}, \ldots, 0\} \to \mathbb{R}^n$ is the vector sequence specifying the initial state of the system, and $\tau(t)$ represents the time-varying delay which is bounded by a positive constant $\tau_{\max}$; this is stated in the following assumption.

Assumption 1 *The delay $\tau : \mathbb{N}_0 \to \mathbb{N}_0$ satisfies*

$$0 \leq \tau(t) \leq \tau_{\max}, \quad t \in \mathbb{N}_0.$$

The system Σ given by (1) is said to be *monotone* if ordered initial states lead to ordered subsequent states. More precisely, let $\boldsymbol{x}(t, \boldsymbol{\varphi})$ denote the solution of Σ with initial state $\boldsymbol{\varphi}(t)$. Then, Σ is monotone if

$$\boldsymbol{\varphi}(t) \leq_\mathcal{K} \boldsymbol{\varphi}'(t), \quad \forall t \in \{-\tau_{\max}, \ldots, 0\},$$

implies that

$$\boldsymbol{x}(t, \boldsymbol{\varphi}) \leq_\mathcal{K} \boldsymbol{x}(t, \boldsymbol{\varphi}'), \quad \forall t \in \mathbb{N},$$

The following result provides a sufficient condition for monotonicity of Σ.

Proposition 1 *If $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, then system Σ given by* (1) *is monotone in $\mathcal{K}$.*

Proof: See Appendix A.

The system Σ given by (1) is called *cone-invariant* with respect to a proper cone $\mathcal{K}$ if its state trajectory starting from any initial state $\boldsymbol{\varphi} \in \mathcal{K}$ will always remain in $\mathcal{K}$, that is $\boldsymbol{x}(t, \boldsymbol{\varphi}) \in \mathcal{K}$ for all $t \in \mathbb{N}_0$. When Σ is cone-invariant with respect to the positive orthant ($\mathcal{K} = \mathbb{R}^n_+$), it is called *positive*. We now provide a necessary and sufficient condition for cone-preservity of monotone systems of the form (1).

Proposition 2 *Assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$. Then, the monotone system Σ given by* (1) *is cone-invariant with respect to $\mathcal{K}$ if and only if*

$$\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}. \tag{2}$$

Proof: See Appendix B.

Note that, according to Proposition 2, if the monotone system (1) has an equilibrium at the origin, *i.e.*,

$$\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0},$$

then it is cone-invariant.

B. Main Results

The following theorem states a sufficient condition for *local* asymptotic stability of the monotone system (1) with time-varying delays satisfying Assumption 1.

Theorem 1 *For the time-delay dynamical system Σ given by* (1)*, suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$. Assume also that there exists a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{v} \in -\textbf{int}\,\mathcal{K}. \tag{3}$$

If $\boldsymbol{x}^\star$ is the only equilibrium point of the cone-invariant monotone system (1) *in*

$$\left\{ x \in \mathcal{K} \ :\ \boldsymbol{0} \leq_\mathcal{K} \boldsymbol{x} \leq_\mathcal{K} \boldsymbol{v} \right\},$$

then for all bounded time-varying delays, $\boldsymbol{x}^\star$ is asymptotically stable with respect to initial conditions satisfying

$$\boldsymbol{0} \leq_\mathcal{K} \boldsymbol{\varphi}(t) \leq_\mathcal{K} \boldsymbol{v}, \quad \forall t \in \{-\tau_{\max}, \ldots, 0\}. \tag{4}$$

Proof: See Appendix C.

Stability condition (3) does not include any information on the magnitude of delays, so it ensures *delay-independent* stability. This type of stability conditions is useful in practice, since the delays may not be easy to model precisely.

We will now show that cone-invariant monotone systems whose vector fields are sub-homogeneous of degree smaller than or equal to one are *globally* asymptotically stable under the stability condition that we have derived in Theorem 1.

Theorem 2 *Assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving and sub-homogeneous of degree $\alpha \in (0,1]$ with respect to $\mathcal{K}$. Furthermore, assume that $\boldsymbol{x}^\star$ is the only equilibrium of* (1) *in $\mathcal{K}$. If there exists a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) - \boldsymbol{v} \in -\textbf{int}\,\mathcal{K}, \tag{5}$$

then the sub-homogeneous cone-invariant monotone system (1) *is globally asymptotically stable for any bounded time-varying delays.*

Proof: See Appendix D.

C. Extensions

Our results can be easily extended to monotone systems with heterogeneous delays of the form:

$$x_i(t+1) = f_i\big(\boldsymbol{x}(t)\big) + g_i\big(x_1(t-\tau_1^i(t)), \ldots, x_n(t-\tau_n^i(t))\big).$$

Here, $i \in \{1,\ldots,n\}$, $\boldsymbol{x}(t) = (x_1(t),\ldots,x_n(t)) \in \mathbb{R}^n$, and $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ldots,f_n(\boldsymbol{x}))$ and $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}),\ldots,g_n(\boldsymbol{x}))$ are order-preserving on a proper cone $\mathcal{K}$. If the delays satisfy

$$0 \leq \tau_i^j(t) \leq \tau_{\max}, \quad \forall i,j.$$

then the stability condition (3) ensures that also this system is locally asymptotically stable.

IV. Continuous-Time Monotone Systems

A. Problem Statement

Next, we consider the continuous-time analog of (1):

$$\mathcal{G}: \begin{cases} \dot{\boldsymbol{x}}(t) & = \boldsymbol{f}\big(\boldsymbol{x}(t)\big) + \boldsymbol{g}\big(\boldsymbol{x}(t-\tau(t))\big), \quad t \geq 0, \\ \boldsymbol{x}(t) & = \boldsymbol{\varphi}(t), \qquad\qquad\qquad\quad t \in [-\tau_{\max}, 0], \end{cases} \tag{6}$$

where $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ are continuously differentiable vector fields on the proper cone $\mathcal{K} \subseteq \mathbb{R}^n$, and the delay $\tau(\cdot)$ is continuous with respect to time, and satisfies

$$0 \leq \tau(t) \leq \tau_{\max}, \quad \forall t \geq 0.$$

System $\mathcal{G}$ given by (6) is called *monotone* if for any initial conditions $\boldsymbol{\varphi}(t), \boldsymbol{\varphi}'(t) \in \mathcal{C}\big([-\tau_{\max},0],\mathcal{K}\big)$, $\boldsymbol{\varphi}(t) \leq_{\mathcal{K}} \boldsymbol{\varphi}'(t)$ for all $t \in [-\tau_{\max}, 0]$ implies that

$$\boldsymbol{x}(t,\boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}'), \quad \forall t \geq 0.$$

Monotonicity of (6) is readily verified using the next result.

Proposition 3 **[2, Theorem 5.1.1]** *Suppose that $\boldsymbol{f}$ is cooperative with respect to a cone $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$. Then, the system $\mathcal{G}$ given by* (6) *is monotone in $\mathcal{K}$.*

System $\mathcal{G}$ is said to be *cone-invariant* with respect to a cone $\mathcal{K}$ if for any initial condition $\boldsymbol{\varphi}(t) \in \mathcal{C}\big([-\tau_{\max},0],\mathcal{K}\big)$, the corresponding state trajectory will never leave $\mathcal{K}$. The following result provides a necessary and sufficient condition for cone-preservity of $\mathcal{G}$.

Proposition 4 *Suppose that $\boldsymbol{f}$ is cooperative with respect to a cone $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$. Then, the monotone system* (6) *is cone-invariant with respect to $\mathcal{K}$ if and only if*

$$\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}. \tag{7}$$

Proof: See Appendix E.

B. Main Results

We now provide a test for the local asymptotic stability of cone-invariant monotone systems of the form (6) with bounded time-varying delays.

Theorem 3 *For the time-delay dynamical system* (6)*, suppose that $\boldsymbol{f}$ is cooperative with respect to a cone $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$. Suppose also that there exist a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v}) \in -\textbf{int}\,\mathcal{K}. \tag{8}$$

If $\boldsymbol{x}^\star$ is the only equilibrium point of the monotone system (6) *such that*

$$\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x}^\star \leq_{\mathcal{K}} \boldsymbol{v},$$

then for initial conditions satisfying

$$\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{\varphi}(t) \leq_{\mathcal{K}} \boldsymbol{v}, \quad \forall t \in [-\tau_{\max}, 0], \tag{9}$$

$\boldsymbol{x}^\star$ is asymptotically stable for any bounded time-varying delays.

Proof: See Appendix F.

Theorem 3 allows us to prove that when a cone-invariant sub-homogeneous monotone system of the form (6) has a unique equilibrium point in the proper cone $\mathcal{K}$, it is globally asymptotically stable under stability condition (8).

Theorem 4 *Assume that $\boldsymbol{f}$ is cooperative with respect to a cone $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$. Furthermore, assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are sub-homogeneous of degree $\alpha > 0$ with respect to $\mathcal{K}$. If $\boldsymbol{x}^\star$ is the only equilibrium of* (6) *in $\mathcal{K}$, and there is a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ satisfying* (8)*, then the sub-homogeneous cone-invariant monotone system $\mathcal{G}$ is globally asymptotically stable for any bounded time-varying delays.*

Proof: The proof is similar to the one of Theorem 2 and thus omitted.

Example 1 *Consider the time-delay system* (6) *with*

$$\begin{aligned} \boldsymbol{f}(x_1,x_2) &= \begin{bmatrix} -\sqrt{2}(x_1+x_2) + \frac{x_1-x_2}{x_1-x_2+2\sqrt{2}} \\ -\sqrt{2}(x_1-x_2) + \frac{x_1+x_2}{x_1+x_2+2\sqrt{2}} \end{bmatrix}, \\ \boldsymbol{g}(x_1,x_2) &= \begin{bmatrix} \frac{\sqrt{2}}{2}(x_1+x_2) \\ -\frac{\sqrt{2}}{2}(x_1-x_2) \end{bmatrix}. \end{aligned} \tag{10}$$

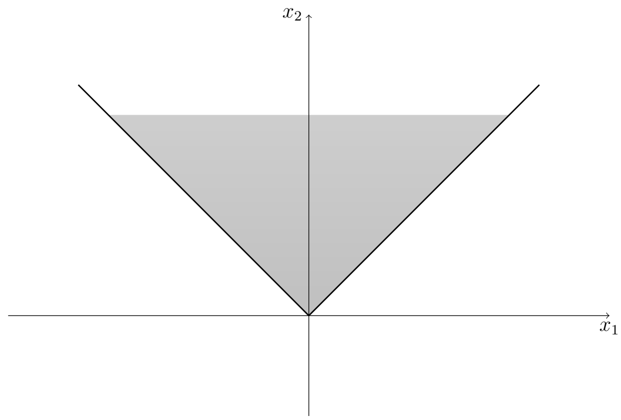

Fig. 1. Cone $\mathcal{K}$ for system (10).

One can verify that (10) *is a sub-homogeneous monotone system and invariant in the cone $\mathcal{K}$, shown in Fig. 1. Moreover, this system has two equilibrium points, one is $\boldsymbol{x}^{\star(1)} = (0,0)$, and the other is $\boldsymbol{x}^{\star(2)} = (0, -\sqrt{2})$. Since the origin is the unique equilibrium in $\mathcal{K}$, it follows from Corollary 4 that for all initial conditions within $\mathcal{K}$ and for any bounded heterogeneous time-varying delays, $\boldsymbol{x}^{\star(1)}$ is globally asymptotically stable.*

C. A Special Case: Cone-invariant Linear Systems

We now discuss delay-independent stability of a special case of (6), namely linear systems on the form

$$\begin{cases} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{x}(t-\tau(t)), \qquad t \geq 0, \\ \boldsymbol{x}(t) &= \boldsymbol{\varphi}(t), \qquad\qquad t \in [-\tau_{\max}, 0]. \end{cases} \tag{11}$$

In terms of (6), $\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$ and $\boldsymbol{g}(\boldsymbol{x}) = B\boldsymbol{x}$. One can verify that if A is cross-positive and B is $\mathcal{K}$-positive, then (11) is a sub-homogeneous cone-invariant system. Theorem 4 helps us to derive a necessary and sufficient condition for delay-independent stability of (11). Specifically, we note the following.

Corollary 1 *Consider the linear system* (11) *where A is cross-positive and B is $\mathcal{K}$-positive. Then, the following statements are equivalent.*

(a) *There exists a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$(A+B)\boldsymbol{v} \in -\textbf{int}\,\mathcal{K}.$$

(b) *The cone-invariant linear system* (11) *is globally asymptotically stable for all bounded time-varying delays.*

Tanaka *et al.* [20] proved that stability of cone-invariant linear delay differential systems is insensitive to arbitrary *constant* time delays using a "DC-dominant" property. More recently, Shen and Zheng [21], by comparing the trajectory of the constant delay system and that of the time-varying delay system, proved that the stability of cone-invariant linear delay differential systems is insensitive to time-varying delays. The latter result is equivalent to Corollary 1.

V. Conclusions and Future Directions

We extended recent delay-independent stability results to sub-homogeneous cone-invariant monotone systems. Specifically, we proved that a sub-homogeneous cone-invariant monotone system is globally asymptotically stable for any bounded heterogeneous time-varying delay if and only if the corresponding delay-free system is globally asymptotically stable. Sub-homogeneous positive monotone systems and linear cone-invariant systems constitute special cases. Illustrative examples demonstrate the validity of our results. Extensions to more general classes of cone-invariant monotone systems, for which the sub-homogeneity assumption does not hold, is part of ongoing research.

Appendix

A. Proof of Proposition 1

Let $\boldsymbol{\varphi}(t)$ and $\boldsymbol{\varphi}'(t)$ be arbitrary initial states satisfying $\boldsymbol{\varphi}(t) \leq_{\mathcal{K}} \boldsymbol{\varphi}'(t)$ for all $t \in \{-\tau_{\max}, \ldots, 0\}$. We show by induction that

$$\boldsymbol{x}(t, \boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}') \tag{12}$$

holds for all $t \in \mathbb{N}$. Since

$$\boldsymbol{x}(0, \boldsymbol{\varphi}) = \boldsymbol{\varphi}(0) \leq_{\mathcal{K}} \boldsymbol{\varphi}'(0) = \boldsymbol{x}(0, \boldsymbol{\varphi}'),$$

the induction hypothesis is true for $t = 0$. Assume for induction that (12) holds for $t \in \{0, \ldots, \hat{t}\}$ with $\hat{t} \in \mathbb{N}_0$. It is clear that $\boldsymbol{x}(\hat{t}, \boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(\hat{t}, \boldsymbol{\varphi}')$. Moreover, as

$$-\tau_{\max} \leq \hat{t} - \tau(\hat{t}) \leq \hat{t},$$

we have $\boldsymbol{x}\big(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}\big) \leq_{\mathcal{K}} \boldsymbol{x}\big(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}'\big)$. It now follows from order-preservity of $\boldsymbol{f}$ and $\boldsymbol{g}$ on $\mathcal{K}$ that

$$\begin{aligned} \boldsymbol{x}\big(\hat{t}+1, \boldsymbol{\varphi}\big) &= \boldsymbol{f}\big(\boldsymbol{x}(\hat{t}, \boldsymbol{\varphi})\big) + \boldsymbol{g}\big(\boldsymbol{x}(\hat{t} - \tau(\hat{t})), \boldsymbol{\varphi})\big) \\ &\leq_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{x}(\hat{t}, \boldsymbol{\varphi}')\big) + \boldsymbol{g}\big(\boldsymbol{x}(\hat{t} - \tau(\hat{t})), \boldsymbol{\varphi}')\big) \\ &= \boldsymbol{x}\big(\hat{t}+1, \boldsymbol{\varphi}'\big). \end{aligned}$$

By induction, we conclude that (12) is true for all $t \in \mathbb{N}_0$. Hence, system (1) is monotone.

B. Proof of Proposition 2

(i) Suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, and that $\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}$. We will prove that system (1) is cone-invariant. Let $\boldsymbol{x}(t, \boldsymbol{\varphi}_0)$ be the solution to (1) with the initial condition $\boldsymbol{\varphi}_0(t) = \boldsymbol{0}$, $t \in \{-\tau_{\max}, \ldots, 0\}$. Clearly,

$$\boldsymbol{x}\big(1, \boldsymbol{\varphi}_0\big) = \boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \in \mathcal{K}.$$

We use induction to show that

$$\boldsymbol{x}\big(t, \boldsymbol{\varphi}_0\big) \in \mathcal{K}, \quad \forall t \in \mathbb{N}. \tag{13}$$

If (13) is true for all t up to some $\hat{t}$, then $\boldsymbol{x}\big(\hat{t}, \boldsymbol{\varphi}_0\big) \in \mathcal{K}$, or, equivalently, $\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x}\big(\hat{t}, \boldsymbol{\varphi}_0\big)$. Also, since $\boldsymbol{\varphi}_0(t) \in \mathcal{K}$ for all $t \in \{-\tau_{\max}, \ldots, 0\}$ and $\hat{t} - \tau(\hat{t}) \in [-\tau_{\max}, \hat{t}]$, we have

$$\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x}\big(\hat{t} - \tau(\hat{t}), \boldsymbol{\varphi}_0\big).$$

As $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$, it follows that

$$\begin{aligned}\mathbf{0} &\leq_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{x}(\hat{t},\boldsymbol{\varphi}_0)\big)+\boldsymbol{g}\big(\boldsymbol{x}(\hat{t}-\tau(\hat{t})),\boldsymbol{\varphi}_0)\big)\\ &= \boldsymbol{x}\big(\hat{t}+1,\boldsymbol{\varphi}_0\big),\end{aligned}$$

implying that (13) holds for all $t \in \mathbb{N}$. Let $\boldsymbol{\varphi}(\cdot)$ be an arbitrary initial state satisfying $\boldsymbol{\varphi}(t) \in \mathcal{K}$, or, equivalently, $\boldsymbol{\varphi}_0(t) \leq_{\mathcal{K}} \boldsymbol{\varphi}(t)$, $t \in \{-\tau_{\max},\dots,0\}$. By Proposition 1, system (1) is monotone and, hence, $\boldsymbol{x}\big(t,\boldsymbol{\varphi}_0\big) \leq_{\mathcal{K}} \boldsymbol{x}\big(t,\boldsymbol{\varphi}\big)$ for all $t \in \mathbb{N}$. It now follows from (13) that $\boldsymbol{x}\big(t,\boldsymbol{\varphi}\big) \in \mathcal{K}$, $\forall t \in \mathbb{N}$, which shows that (1) is cone-invariant.

(ii) Assume that the monotone system (1) is cone-invariant. If, for contradiction, $\boldsymbol{f}(\mathbf{0})+\boldsymbol{g}(\mathbf{0}) \notin \mathcal{K}$, then there exists $\boldsymbol{z} \in \mathcal{K}^{\star}$ such that $\boldsymbol{z}^{\top}\big(\boldsymbol{f}(\mathbf{0})+\boldsymbol{g}(\mathbf{0})\big) < 0$. Thus, $z^{\top}\boldsymbol{x}\big(1,\varphi_0\big) = \boldsymbol{z}^{\top}\big(\boldsymbol{f}(\mathbf{0})+\boldsymbol{g}(\mathbf{0})\big) < 0$, implying that $\boldsymbol{x}\big(1,\varphi_0\big) \notin \mathcal{K}$. This contradicts the fact that (1) is cone-invariant.

C. Proof of Theorem 1

Before proving Theorem 1, we state the lemma that is the key to our argument.

Lemma 1 *Consider the following time-delay dynamical system with constant delays, closely related to system* (1)*:*

$$\Sigma' : \begin{cases} \boldsymbol{y}\big(t+1\big) &= \boldsymbol{f}\big(\boldsymbol{y}(t)\big)+\boldsymbol{g}\big(\boldsymbol{y}(t-\tau_{\max})\big), t \in \mathbb{N}_0,\\ \boldsymbol{y}\big(t\big) &= \boldsymbol{\varphi}\big(t\big), \qquad\qquad t \in \{-\tau_{\max},\dots,0\}.\end{cases} \tag{14}$$

Assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$. The following statements hold.

(i) *If there exists a vector $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ such that*

$$\boldsymbol{f}(\boldsymbol{v})+\boldsymbol{g}(\boldsymbol{v})-\boldsymbol{v} \in -\textbf{int}\,\mathcal{K}, \tag{15}$$

then

$$\boldsymbol{y}(t+1,\boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{y}(t,\boldsymbol{\varphi}_v), \quad t \in \mathbb{N}_0, \tag{16}$$

$$\boldsymbol{x}(t,\boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{y}(t,\boldsymbol{\varphi}_v), \quad t \in \mathbb{N}_0, \tag{17}$$

where $\boldsymbol{\varphi}_v(t) = \boldsymbol{v}$ for all $t \in \{-\tau_{\max},\dots,0\}$, and $\boldsymbol{x}(t,\boldsymbol{\varphi}_v)$ and $\boldsymbol{y}(t,\boldsymbol{\varphi}_v)$ are solutions to (1) *and* (14)*, respectively.*

(ii) *If $\boldsymbol{f}(\mathbf{0})+\boldsymbol{g}(\mathbf{0}) \in \mathcal{K}$, then the solution $\boldsymbol{y}(t,\boldsymbol{\varphi}_0)$ to* (14) *starting from $\boldsymbol{\varphi}_0(t)=0$, $t \in \{-\tau_{\max},\dots,0\}$, satisfies*

$$\begin{aligned}\boldsymbol{y}(t,\boldsymbol{\varphi}_0) &\leq_{\mathcal{K}} \boldsymbol{y}(t+1,\boldsymbol{\varphi}_0), \quad t \in \mathbb{N}_0,\\ \boldsymbol{y}(t,\boldsymbol{\varphi}_0) &\leq_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}_0), \quad t \in \mathbb{N}_0,\end{aligned}$$

where $\boldsymbol{x}(t,\boldsymbol{\varphi}_0)$ is the solution to (1)*.*

Proof:

(i) Let $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ be a vector satisfying (15). Since

$$\boldsymbol{y}\big(1,\boldsymbol{\varphi}_v\big) = \boldsymbol{f}(\boldsymbol{v})+\boldsymbol{g}(\boldsymbol{v}) \leq_{\mathcal{K}} \boldsymbol{v} = \boldsymbol{y}\big(0,\boldsymbol{\varphi}_v\big),$$

relation (16) holds for $t=0$. Assume that (16) is true for all t up to $\hat{t}$. It follows from the induction hypothesis that

$$\begin{aligned}\boldsymbol{y}(\hat{t}+1,\boldsymbol{\varphi}_v) &\leq_{\mathcal{K}} \boldsymbol{y}(\hat{t},\boldsymbol{\varphi}_v),\\ \boldsymbol{y}(\hat{t}-\tau_{\max}+1,\boldsymbol{\varphi}_v) &\leq_{\mathcal{K}} \boldsymbol{y}(\hat{t}-\tau_{\max},\boldsymbol{\varphi}_v).\end{aligned}$$

These inequalities together with the order-preservity of $\boldsymbol{f}$ and $\boldsymbol{g}$ imply that

$$\begin{aligned}\boldsymbol{y}\big(\hat{t}+2,\boldsymbol{\varphi}_v\big) &= \boldsymbol{f}\big(\boldsymbol{y}(\hat{t}+1,\boldsymbol{\varphi}_v)\big)+\boldsymbol{g}\big(\boldsymbol{y}(\hat{t}-\tau_{\max}+1,\boldsymbol{\varphi}_v)\big)\\ &\leq_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{y}(\hat{t},\boldsymbol{\varphi}_v)\big)+\boldsymbol{g}\big(\boldsymbol{y}(\hat{t}-\tau_{\max},\boldsymbol{\varphi}_v)\big)\\ &= \boldsymbol{y}\big(\hat{t}+1,\boldsymbol{\varphi}_v\big).\end{aligned}$$

Therefore, (16) holds for all $t \in \mathbb{N}_0$.

By using induction, we now prove (17). The induction hypothesis is true for $t=1$, since

$$\boldsymbol{x}\big(1,\boldsymbol{\varphi}_v\big) = \boldsymbol{f}(\boldsymbol{v})+\boldsymbol{g}(\boldsymbol{v}) = \boldsymbol{y}\big(1,\boldsymbol{\varphi}_v\big).$$

Assuming it is true for a given $t=\hat{t}$, we then have

$$\begin{aligned}\boldsymbol{x}(\hat{t},\boldsymbol{\varphi}_v) &\leq_{\mathcal{K}} \boldsymbol{y}(\hat{t},\boldsymbol{\varphi}_v),\\ \boldsymbol{x}(\hat{t}-\tau(\hat{t}),\boldsymbol{\varphi}_v) &\leq_{\mathcal{K}} \boldsymbol{y}(\hat{t}-\tau(\hat{t}),\boldsymbol{\varphi}_v).\end{aligned}$$

As $\hat{t}-\tau_{\max} \leq \hat{t}-\tau(\hat{t})$, it follows from (17) that

$$\boldsymbol{x}(\hat{t}-\tau(\hat{t}),\boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{y}(\hat{t}-\tau_{\max},\boldsymbol{\varphi}_v),$$

implying that

$$\begin{aligned}\boldsymbol{x}\big(\hat{t}+1,\boldsymbol{\varphi}_v\big) &= \boldsymbol{f}\big(\boldsymbol{x}(\hat{t},\boldsymbol{\varphi}_v)\big)+\boldsymbol{g}\big(\boldsymbol{x}(\hat{t}-\tau(\hat{t}),\boldsymbol{\varphi}_v)\big)\\ &\leq_{\mathcal{K}} \boldsymbol{f}\big(\boldsymbol{y}(\hat{t},\boldsymbol{\varphi}_v)\big)+\boldsymbol{g}\big(\boldsymbol{y}(\hat{t}-\tau_{\max},\boldsymbol{\varphi}_v)\big)\\ &= \boldsymbol{y}\big(\hat{t}+1,\boldsymbol{\varphi}_v\big),\end{aligned}$$

where the inequality follows from the fact that $\boldsymbol{f}$ and $\boldsymbol{g}$ are order-preserving on $\mathcal{K}$. The induction proof is complete.

(ii) The proof is similar to the one of part *(i)* and thus omitted. ∎

Proof of Theorem 1:

Let $\boldsymbol{v}$ be a vector such that (3) holds. System (1) is monotone and cone-invariant. Thus, $\boldsymbol{f}(\mathbf{0})+\boldsymbol{g}(\mathbf{0}) \in \mathcal{K}$. Define $\boldsymbol{\varphi}_0(t) = \mathbf{0}$ and $\boldsymbol{\varphi}_v(t) = \boldsymbol{v}$, $t \in \{-\tau_{\max},\dots,0\}$. For any initial condition $\boldsymbol{\varphi}(t)$ satisfying (5), since (3) is monotone, we have

$$\boldsymbol{x}(t,\boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}_v), \quad \forall t\mathbb{N}_0.$$

Let $\boldsymbol{y}(t,\boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t,\boldsymbol{\varphi}_v)$ be solutions to (14) starting from $\boldsymbol{\varphi}_0(t)$ and $\boldsymbol{\varphi}_v(t)$, respectively. According to Lemma 1, $\boldsymbol{y}(t,\boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}_0)$ and $\boldsymbol{x}(t,\boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{y}(t,\boldsymbol{\varphi}_v)$ for all $t \in \mathbb{N}_0$, implying that

$$\boldsymbol{y}(t,\boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t,\boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{y}(t,\boldsymbol{\varphi}_v), \quad \forall t \in \mathbb{N}_0. \tag{18}$$

Moreover, by Lemma 1, $\boldsymbol{y}(t,\boldsymbol{\varphi}_0)$ is non-decreasing and $\boldsymbol{y}(t,\boldsymbol{\varphi}_v)$ is non-increasing for $t \in \mathbb{N}_0$, which implies that

$$\mathbf{0} \leq_{\mathcal{K}} \boldsymbol{y}(t,\boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{y}(t,\boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{v}, \quad \forall t \in \mathbb{N}_0.$$

Thus, both $\boldsymbol{y}(t,\boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t,\boldsymbol{\varphi}_v)$ are bounded and monotone. It now follows from [2, Theorem 1.2.1] that $\boldsymbol{y}(t,\boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t,\boldsymbol{\varphi}_v)$ converge to an equilibrium of (1) in

$$\big\{x \in \mathcal{K} \;:\; \mathbf{0} \leq_{\mathcal{K}} \boldsymbol{x} \leq_{\mathcal{K}} \boldsymbol{v}\big\},$$

which must be $\boldsymbol{x}^{\star}$, *i.e.*,

$$\lim_{t\to\infty} \boldsymbol{y}(t,\boldsymbol{\varphi}_0) = \lim_{t\to\infty} \boldsymbol{y}(t,\boldsymbol{\varphi}_v) = \boldsymbol{x}^{\star}. \tag{19}$$

It follows from (18) and (19) that $\lim_{t\to\infty} \boldsymbol{x}(t,\boldsymbol{\varphi}) = \boldsymbol{x}^{\star}$. This completes the proof.

D. Proof of Theorem 2

Let $\boldsymbol{v} \in \textbf{int}\,\mathcal{K}$ be a vector satisfying (5) and $\boldsymbol{\varphi}(\cdot) \in \mathcal{K}$ be an arbitrary initial condition. For all $\boldsymbol{z} \in \mathcal{K}^\star \backslash \{\boldsymbol{0}\}$, we have $\boldsymbol{z}^\top \boldsymbol{v} > 0$. Thus, we can choose $\gamma \geq 1$ such that $\boldsymbol{z}^\top \boldsymbol{\varphi}(t) \leq \gamma \boldsymbol{z}^\top \boldsymbol{v}$, for $t \in \{-\tau_{\max}, \dots, 0\}$, which implies that

$$\boldsymbol{\varphi}(t) \leq_{\mathcal{K}} \gamma \boldsymbol{v}. \tag{20}$$

Moreover, since $\boldsymbol{f}$ and $\boldsymbol{g}$ are sub-homogeneous of degree $\alpha \in (0, 1]$,

$$\begin{aligned}\boldsymbol{f}(\gamma\boldsymbol{v}) + \boldsymbol{g}(\gamma\boldsymbol{v}) &\leq_{\mathcal{K}} \gamma^\alpha \big(\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v})\big) \\ &\leq_{\mathcal{K}} \gamma \big(\boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v})\big) \leq_{\mathcal{K}} \gamma \boldsymbol{v},\end{aligned}$$

where the last inequality follows from (5). Therefore, the vector $\gamma\boldsymbol{v}$ also satisfies (5). It follows from Theorem 1 and (20) that $\lim_{t\to\infty} \boldsymbol{x}(t, \boldsymbol{\varphi}) = \boldsymbol{x}^\star$.

E. Proof of Proposition 4

(i) Let $\boldsymbol{\varphi}_0(t)$ be the initial condition satisfying $\boldsymbol{\varphi}_0(t) = \boldsymbol{0}$, $t \in [-\tau_{\max}, 0]$. Since $\boldsymbol{f}$ is cooperative with respect to $\mathcal{K}$ and $\boldsymbol{g}$ is order-preserving on $\mathcal{K}$, it follows from Proposition 3 that system (6) is monotone. Thus, if $\boldsymbol{\varphi}_0(t) \leq_{\mathcal{K}} \boldsymbol{\varphi}(t)$ for all $t \in [-\tau_{\max}, 0]$, then

$$\boldsymbol{x}(t, \boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}), \quad \forall t \geq 0. \tag{21}$$

Let $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ be the solution to the following delayed differential equation with constant delays:

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \boldsymbol{f}\big(\boldsymbol{y}(t)\big) + \boldsymbol{g}\big(\boldsymbol{y}(t - \tau_{\max})\big), & t \geq 0, \\ \boldsymbol{y}(t) = \boldsymbol{\varphi}(t), & t \in [-\tau_{\max}, 0]. \end{cases} \tag{22}$$

According to [2, Corollary 5.2.2], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ is non-decreasing, *i.e.*, $\boldsymbol{0} = \boldsymbol{\varphi}_0(0) \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_0), \quad \forall t \geq 0$. Moreover, by [19, Lemma 2], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}_0)$ for all $t \geq 0$. Therefore, $\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi})$, which implies that $\boldsymbol{x}(t, \boldsymbol{\varphi}) \in \mathcal{K}$ for all $t \geq 0$.

Conversely, assume that (6) is cone-invariant. Suppose, for contradiction, that $\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0}) \notin \mathcal{K}$. Then, there is $\boldsymbol{z} \in \mathcal{K}^\star$ such that $\boldsymbol{z}^\top \big(\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0})\big) < 0$, implying that $D^+ \boldsymbol{z}^\top \boldsymbol{x}(0, \boldsymbol{\varphi}_0) = \boldsymbol{z}^\top \big(\boldsymbol{f}(\boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{0})\big) < 0$. Hence, there exists sufficiently small $\delta > 0$ such that

$$\boldsymbol{z}^\top \boldsymbol{x}(t, \boldsymbol{\varphi}_0) < \boldsymbol{z}^\top \boldsymbol{x}(0, \boldsymbol{\varphi}_0) = 0, \quad \forall t \in (0, \delta).$$

Therefore, $\boldsymbol{x}(t) \notin \mathcal{K}$ for $t \in (0, \delta)$, which is a contradiction.

F. Proof of Theorem 3

Let $\boldsymbol{v}$ be a vector satisfying (8). Define $\boldsymbol{\varphi}_0(t) = \boldsymbol{0}$ and $\boldsymbol{\varphi}_v(t) = \boldsymbol{v}$, $t \in [-\tau_{\max}, 0]$. As $\boldsymbol{f}$ is cooperative and $\boldsymbol{g}$ is order-preserving, according to Proposition 3, system (6) is monotone. Thus, $\boldsymbol{x}(t, \boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}_v)$, $\forall t \geq 0$. Let $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t, \boldsymbol{\psi}_v)$ be solutions to system (22) starting from $\boldsymbol{\varphi}_w(t)$ and $\boldsymbol{\varphi}_v(t)$, respectively. According to [19, Lemma 2], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{x}(t, \boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ for all $t \geq 0$, implying that

$$\boldsymbol{y}(t, \boldsymbol{\varphi}_w) \leq_{\mathcal{K}} \boldsymbol{x}(t, \boldsymbol{\varphi}) \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v), \quad \forall t \geq 0. \tag{23}$$

Since $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ is non-decreasing and $\boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ is non-increasing for $t \geq 0$, we have

$$\boldsymbol{0} \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_w) \leq_{\mathcal{K}} \boldsymbol{y}(t, \boldsymbol{\varphi}_v) \leq_{\mathcal{K}} \boldsymbol{v}, \quad \forall t \geq 0.$$

Thus, both $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ are bounded and monotone. Therefore, by [2, Theorem 1.2.1], $\boldsymbol{y}(t, \boldsymbol{\varphi}_0)$ and $\boldsymbol{y}(t, \boldsymbol{\varphi}_v)$ converge to an equilibrium of (22) in $[\boldsymbol{w}, \boldsymbol{v}]$, which must be $\boldsymbol{x}^\star$, *i.e.*,

$$\lim_{t\to\infty} \boldsymbol{y}(t, \boldsymbol{\varphi}_0) = \lim_{t\to\infty} \boldsymbol{y}(t, \boldsymbol{\varphi}_v) = \boldsymbol{x}^\star. \tag{24}$$

It now follows from (23) and (24) that $\lim_{t\to\infty} \boldsymbol{x}(t, \boldsymbol{\varphi}) = \boldsymbol{x}^\star$.

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